THE PRINCIPLES

OF

FLUXIONS.

THE

PRINCIPLES OF FLUXIONS:

DESIGNED FOR THE

USE OF STUDENTS

IN THE

Universities.

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SECOND EDITION,

WITH CORRECTIONS AND CONSIDERABLE ADDITIONS.

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PREFACE.

The greater part of the following Work was drawn up by the Author, when engaged in a public situation in the University, chiefly for the benefit of his own pupils. Under the hope that it may be useful to others, he has endeavoured to render it as complete as the nature of his plan appeared to admit; and he now submits it to the judgment of the public.

Should this Book fall into the hands of any who have made considerable advances in pure Mathematics, they will perhaps find little to repay the labour of perusal. Few however of those, who have entered upon mathematical studies during their residence in the University, continue the pursuit after their first degree; and far the greater part are contented with such a portion of analytical knowledge, as may suffice to illustrate the chief Propositions in Newton's *Principia*. It is for this class of students that the following Treatise is intended: and if it be found useful by them, the object of publication is answered.

In preparing these sheets for the Press, the Author proposed to himself two rules, from which he has not intentionally deviated.

The first rule was, to illustrate every thing in the simplest manner.

Many are deterred from the study of Fluxions by the apparent abstruseness and difficulty of the science: and of those, who by the system of University Lectures are induced to undertake it, comparatively few have resolution to proceed. It is highly important that every help should be afforded; and in this, as well as in every other, department of literature, those persons are well employed, who by facilitating the attainment of knowledge increase the number of students, and thus add to the general stock of intellectual improvement.

In pursuance of this plan, the Author has on most occasions begun with the simplest instance of the application of fluxional principles, and has then proceeded to the more general cases. The custom of deducing conclusions in particular instances from general expressions, however useful in practice, does not furnish the mode, by which Fluxions can be made easy to a beginner. He should be led on by degrees from the most familiar to the general propositions. Every step will thus be intelligible; and he will have double the advantage of increasing his knowledge, and at the

same time of improving by the best exercise his intellectual powers.

For it must not be forgotten, that one of the great benefits to be derived from mathematical studies is the discipline of the mind. The mere knowledge of certain truths is, to the great body of literary men, a matter only of secondary importance, when compared with the advantages, which result· from the exercise of the understanding, and the improvement of the reasoning faculty. The Elements of Euclid have in this view been justly considered as of singular excellence. Their peculiar value arises in a great measure from the perspicuity of every part. The chain of reasoning is preserved entire; and the reader proceeds from step to step with the argument fully before him, and with an evidence of its truth which cannot be doubted.

It were to be wished, that all elementary books might, as far as possible, be composed upon this principle. Abstruseness is never to be affected for its own sake; and it scarcely can be affected by those, who regard the benefit of others as the end of their labour.

The method of Fluxions rests upon a principle purely analytical; namely, the theory of limiting ratios; and the subject may therefore be considered as one of pure mathematics, without any regard to ideas of time and velocity. But the usual manner of treating it is to employ con-

siderations resulting from the theory of motion. This was the plan of Sir Isaac Newton in first delivering the principles of the method; and it is adopted in the following Work, from the belief, that it is well adapted for illustration, and calculated to give the greatest facilities to the Student.

The second rule, which has been observed in this Treatise, was to introduce every subject which an ordinary student is likely to require.

With this intention, the Author has freely availed himself of former publications. While he, has carefully avoided every thing abstruse, and which did not seem to fall within his plan, he has introduced, as he believes, many articles, which were not easily accessible, on account of the scarcity of the works which contained them. Some of the Propositions of Cotes and De Moivre are in frequent use; but their works are not always to be procured.

In presenting to the public a second Edition, the Author feels himself bound to express his gratitude, for the manner in which the first was received. He is happy to acknowledge his obligation to those, who have done him the honor to recommend the Work: and his thanks are especially due to Dr. Hutton, for the very favourable notice, which he was pleased to take of it in his "Course of Mathematics."

Many suggestions have been offered by the Reviewers, in the spirit of liberal and candid criticism, for the improvement of the Treatise. The Author certainly admits the justice and feels the kindness of their remarks ; and he readily owns that several subjects are excluded, which ought to be found in any Treatise professing to be complete. But he must beg permission again to state, that his Work does not claim to be of that description : it was meant for the use of those, who " are contented " with such a portion of analytical knowledge, as " may suffice to illustrate the chief Propositions in " Newton's *Principia :*" and although he has in deference to the judgment of his candid Reviewers introduced two additional Chapters, taken from Cotes and Hayes, yet it did not appear to him that he could enter upon certain other subjects, without a material deviation from his original plan. In an undertaking of this sort, his necessary occupations forbid him to engage : and he indulges the hope that the publication will be found useful by many, in its present form.

Clapham,
March 4, 1816.

FLUXIONS.

CHAP. I.

TO FIND THE FLUXIONS OF QUANTITIES.

Art. (1.) QUANTITIES of all kinds are here considered as produced by motion. Thus, if a point *A* be conceived to move in the direction *AB*, it will trace out the line *AB*. If *AB* move parallel to itself, it will trace out a parallelogram, as *ABCD*. If this parallelogram be supposed to move in a direction perpendicular to its plane, it will generate a parallelopiped, as *AGFB*. Hence solids are conceived to be generated by the motion of surfaces; surfaces by the motion of lines; and lines by the motion of points.
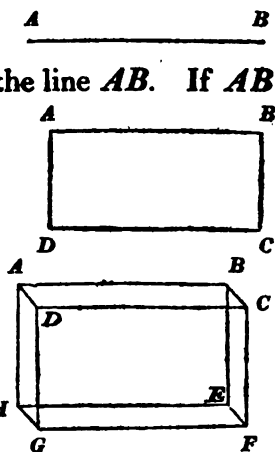
(2.) The quantity which is thus generated is called the *fluent*, or the *flowing* quantity.

(3.) The fluxion of a quantity at any point of time is its increment or decrement, taken proportional to the velocity with which the quantity flows at that time.

Cor. A constant quantity has no fluxion.

(4.) The velocities by which these quantities are produced may be either uniform or variable. If the velocities of two points A and C be uniform, and in the ratio of 1 : 2, the lines AB and CD, described in the same time, will be in the same ratio of 1 : 2 ; and in this case the line CD is said to *flow* with twice the velocity of AB; the increase of CD in a given time is double of the increase of AB in the same time, or the fluxion of CD : the fluxion of AB :: 2 : 1. And in general, if the uniform velocity of the point A : the uniform velocity of the point C :: m : n, the corresponding increments of AB and CD will be in the same ratio ; that is, the fluxion of AB : the fluxion of CD :: m : n or :: velocity of A : velocity of C.
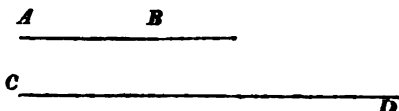
If the lines AB and CD are described with variable velocities, their corresponding increments are no longer
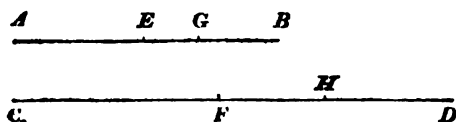
proportional to the velocities, and therefore cannot represent the fluxions. The velocity of a body at any point of its motion is represented by the space, which would be described in a given time with the velocity at that point continued uniform. Let E and F be two contemporaneous positions of A and C; let EG and FH represent two spaces, which A and C would describe in the same time, if the velocities at E and at F were continued uniformly ; and let EB and FD be the spaces which they actually describe by the variable velocities. The parts GB and HD are produced by accelerations which had no existence at E and F, and are not described uniformly ;

whereas EG and FH are proportional to the velocities at E and F, and are described in the same time. Hence the fluxion of AE : fluxion of CF :: EG : FH.

(5.) This reasoning may be illustrated by the doctrine of falling bodies. If a body descend by the force of gravity for $2''$, it describes a space in the 2d second of 48 feet, nearly; this may be divided into two parts; of which 32 feet are described by the velocity acquired at the end of the 1st second, and 16 by the acceleration of gravity. If then we should assume 48 as the measure of the velocity in the 2d second, the conclusion would be erroneous. Suppose a body to fall for $10''$, the space described in the 11th second may be thus represented.

The velocity acquired in $10''$, omitting fractions,
$$= 320 \text{ feet;}$$
or the space described in the 11th second by   Feet.
  that uniform velocity . . . . . . $= 320$
  The space in the 11th second, by gravity $= \quad 16$

∴ the whole space, from both causes . . $= 336$

On this division of the spaces depends the whole method of fluxions.

(6.) When a quantity increases with a velocity which continually varies, the quantity, which measures the fluxion, is a limit between the preceding and succeeding increments, and is ultimately * equal to either of them.

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* The word *ultimately* is intended to denote that particular instant, when the time is diminished *sine limite.* Sir ISAAC NEWTON thus describes ultimate velocity and ultimate ratios: " Per velocitatem ultimam intelligi eam, quâ corpus movetur, neque antequam attingit locum ultimum et motus cessat, neque postea, sed tunc cum attingit; id est, illam ipsam velocitatem quâcum corpus attingit locum ultimum,

et

1. Let the point A move on the straight line AB with

$$A \qquad\qquad F\ C \quad B \quad E \quad\ D$$
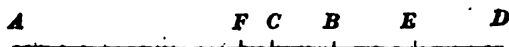
a velocity perpetually increasing. Let CB represent the space described in a given time before A arrives at B; and let BD be the space described in the same time afterwards; both by the variable velocity. Suppose BE to be the space which the point A would describe in the same time, with the velocity at B continued uniform; and in the line BA take $BF = BE$. Then, since the velocity of A is perpetually increasing, BE is less than BD, and greater than BC; but BE measures the fluxion; therefore the quantity which measures the fluxion is in this case greater than the preceding, and less than the succeeding increment.

Next, let the time of describing CB or BD be assumed extremely small; the difference between BF and BC, or between BE and BD, will ultimately be less than any assignable magnitude. For ED is described by an acceleration, which was nothing at B; and BE is described by an uniform finite velocity; therefore, by diminishing the time *sine limite*, ED is indefinitely diminished with respect to BE, and FC with respect to BF; or BE is ultimately equal to BD, and BC to BF; that is, the quantity which measures the fluxion is a limit between the increments, and ultimately equal either to the preceding or succeeding increment.

2. Let AB be traced out by a velocity perpetually

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et quâcum motus cessat. Et similiter per ultimam rationem quantitatum evanescentium, intelligendam esse rationem quantitatum, non antequam evanescunt, non postea, sed quâcum evanescunt." Scholium Sect. Primæ.

retarded. In this case, BC, the preceding increment,

A   C   F   B   D   E

is greater than BF or BE, and BD is less; the reason-
ing and the conclusion are the same as before.

COR. 1. If one quantity, as AB, increase uniformly,
and another, as CD, in-
crease with an accele-
rated or retarded velo-
city; the fluxion of
AB : fluxion of CD :: increment of AB in a given
time : limit between the corresponding increments of CD.
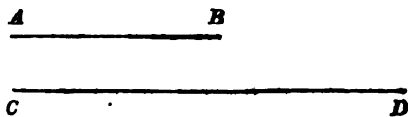
COR. 2. Hence the fluxion of a quantity must have
these two properties. 1. It must be a limit between
the preceding and succeeding increments. 2. It must
be proportional to the increment of a quantity, which
flows uniformly.
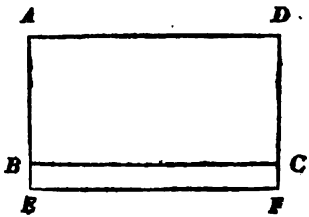
(7.) In the preceding demonstration we have reasoned
upon the hypothesis, that if the time be indefinitely
small, DE vanishes in respect of BE. This may be
illustrated in the following manner. Suppose a pro-
jectile to be discharged in a direction GH perpendicular
to the horizon, and with a velocity of 1000 feet
in a second, the force of gravity being considered
as uniform, and the resistance of the air omitted.
Take $GL = 1000$ feet, $LM = 16\frac{1}{13}$; then, at the
end of $1''$, the body would be found at M, and
$LM : LG :: 16 : 1000$ nearly :: $2 : 125$. Next,
take $GN = 500$ feet, and $NO = 4$ feet. Then, in
half a second, while the body would uniformly
describe GN, gravity would draw it through NO;
therefore it will be found at O; and $NO : NG$
:: $4 : 500 :: 1 : 125$. Next, take $GP = 100$; then

Pv, the space by gravity in $\frac{1}{10}''$, or whilst the body with an uniform motion would describe 100 feet, $= \frac{4}{25}$ of a foot; therefore in this case $Pv : PG :: \frac{4}{25} : 100 :: 4 : 2500 :: 1 : 625$. If we take the one thousandth part of a second, $Pv : PG :: 1 : 62,500$, nearly; so that in every case, as the time is diminished, the space through which gravity would draw the body bears a less ratio to the space described in that same time by the uniform finite velocity. And if the time be assumed indefinitely small, Pv will ultimately become evanescent in respect of PG.

(8.) The first letters of the alphabet, a, b, c, d, &c. are generally assumed to represent invariable quantities; the last letters, as w, x, y, z, such as are variable. The fluxion of a simple quantity x is denoted by a point over the letter, as $\dot{x}$. Thus, if AB is equal to x, BC, the fluxion of AB, $= \dot{x}$.

(9.) If the fluxion of x be expressed by $\dot{x}$, the fluxion of ax will be $a\dot{x}$.

For if x increase uniformly, ax will also increase uniformly, and its velocity of increase will be a times greater than that of x, that is, it will $= a\dot{x}$. Let $ABCD$ be a parallelogram, of which the side AD remains constant, while AB flows uniformly. $BCFE$ will represent its fluxion. Let $AD = a$, $AB = x$, $BE = \dot{x}$; then $ABCD = ax$, and $BCFE = a\dot{x}$.

(10.) If two quantities have to each other a given ratio, their fluxions are in the same given ratio.

Let $x : y :: a : b,$

then $bx = ay$; $\therefore b\dot{x} = a\dot{y}$;

$\therefore \dot{x} : \dot{y} :: a : b.$

(11.) The fluxion of $a \pm x$ is $\pm \dot{x}$.

Let

AB, a constant line, $= a,$

$CD,$ a variable line, $= x.$

The line AB does not affect the increase or decrease of x; so that the variation of $AB + CD$ is the same as the variation of CD alone; that is, the fluxion of $a \pm x = \pm \dot{x}.$

COR. Constant quantities, connected with variable ones by the sign $+$ or $-$, disappear when the fluxion is taken.

(12.) If any numerical or algebraical quantity x be supposed to increase uniformly, the squares of the succeeding quantities will increase with velocities continually accelerated.

Let the numbers 5, 6, 7, 8, 9, 10, be assumed, which increase uniformly, as the several differences $= 1$. Their squares are 25, 36, 49, 64, 81, 100, of which the several differences are 11, 13, 15, 17, 19; therefore the numbers do not increase uniformly.

In general, let x represent any algebraical quantity, and x' its increment; then the present and succeeding values are $x, x + x', x + 2x', x + 3x'$, &c.; and the present and succeeding values of the squares are x^2, $(x + x')^2$, $(x + 2x')^2$, $(x + 3x')^2$, &c.

The increments are obtained by subtracting in this

case each preceding value from that which follows it; therefore the increments are, $2xx' + x'^2$, $2xx' + 3x'^2$, $2xx' + 5x'^2$ &c.; that is, if the quantities themselves increase with an uniform velocity, their squares increase with a velocity perpetually accelerated. The same reasoning is true of the higher powers.

Cor. It is manifest, that if $2xx'$ were equal to the difference between these several squares, their velocity of increase would be uniform; so that the parts x'^2, $3x'^2$, $5x'^2$, &c. are the effects of acceleration; whence (by Art. 3. and 4.) these quantities, which involve the powers of x', are to be omitted in taking the fluxions.

(13.) To find the fluxion of x^2.

Let $x - x'$, x, and $x + x'$, represent the preceding, present, and succeeding values of x. Their squares are

$$x^2 - 2xx' + x'^2, \quad x^2, \quad \text{and} \quad x^2 + 2xx' + x'^2;$$

∴ the preceding increment $= 2xx' - x'^2$, and the succeeding . . . . . . $= 2xx' + x'^2$, of which the limit is $2xx'$, or the fluxion $= 2x\dot{x}$.

As an illustration of the problem, let $AB = x$; $ABCD = x^2$; let AE and AF each $=$ the increment of $x = x'$; and let $EGHD$, $FKMD$ represent the preceding and succeeding values of $ABCD$. Then the gnomon APH, or the preceding increment, $= AP + BH - BG = 2xx' - x'^2$; and the succeeding increment, $FLC = 2xx' + x'^2$. Now the limit between these is $2xx'$, or ultimately $2x\dot{x}$; ∴ the fluxion of $x^2 = 2x\dot{x}$.
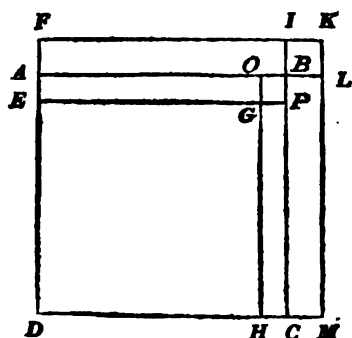
(14.) To find the fluxion of a rectangle xy.

$$(x+y)^2 = x^2 + 2xy + y^2.$$

Let $x+y = z$ $\}$ then $(x+y)^2 = z^2$; $\therefore$ the fluxion of
$\therefore \dot{x} + \dot{y} = \dot{z}$ $\}$

$(x+y)^2 =$ the fluxion of $z^2 = 2z\dot{z}$, or $= 2 (x+y) (\dot{x}+\dot{y}) = 2x\dot{x} + 2x\dot{y} + 2y\dot{x} + 2y\dot{y}$; but the fluxion of $(x+y)^2 =$ the fluxion of $x^2 + 2xy + y^2 = 2x\dot{x} +$ the fluxion of $2xy + 2y\dot{y}$;

$\therefore 2x\dot{x} + 2x\dot{y} + 2y\dot{x} + 2y\dot{y} = 2x\dot{x} +$ the flux. of $2xy + 2y\dot{y}$;

$\therefore 2x\dot{y} + 2y\dot{x} =$ the fluxion of $2xy$, or $= 2 \times$ fluxion of xy;

$\therefore x\dot{y} + y\dot{x} =$ the fluxion of xy.

Illustration. Let $ABCD$ be the rectangle; $AD = x$, $AB = y$. Take AE, AH, each $= x'$, and BN, BM, each $= y'$, and complete the rectangles $EFGD, HIKD$.
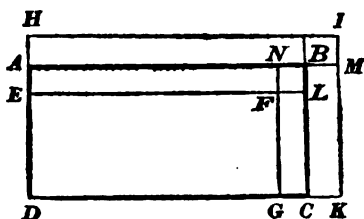
These two rectangles will represent the preceding and succeeding values of $ABCD$. The preceding increment $=$ the gnomon $ALG = AL + BG - BF$,

or . . . $=$ . . . $yx' + xy' - y'x'$;

and the succeeding increment $= HB + BK + IB$,

or . . . $=$ . . . $yx' + xy' + y'x'$;

and the limit between them is $yx' + xy'$, or the fluxion is $y\dot{x} + x\dot{y}$.

RULE. The fluxion of the product of two flowing quantities is equal to the sum of the product of each quantity and the fluxion of the other.

(15.) In this Article, x and y are both supposed to increase; for $\dot{x}$ and $\dot{y}$ are both assumed positive. But if one of them, as x, decrease, whilst the other continues to increase, the fluxion is $x\dot{y} - y\dot{x}$.

Let *ABCD* be a parallelo-gram, of which the side *AB* increases, and *AD* decreases, with an uniform velocity; take $AB = y$, $AD = x$; $BG = \dot{y}$, $DE = \dot{x}$; and by this variation of the sides, let *ABCD* be changed into *AEHG*. The fluxion of $ABCD = BH - EC = BO - EO = x\dot{y} - \dot{x}$ $(y + \dot{y}) = x\dot{y} - y\dot{x} - \dot{x}\dot{y} =$ ultimately, $x\dot{y} - y\dot{x}$.
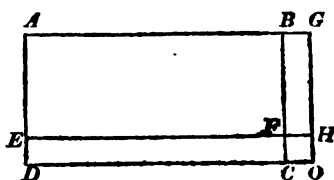
Hence, to express the rate at which any quantity in-creases, the fluxion of the part which increases must be written with the sign $+$, and that which decreases with the sign $-$.

If a negative quantity increase, it must be considered as a decreasing positive quantity, and its fluxion is $-$.

(16.) To find the fluxion of xyz.
Let $xy = v$; then $xyz = vz$; $\therefore$ the fluxion of $xyz =$ the fluxion of $vz = v\dot{z} + z\dot{v}$; but $\dot{v} = x\dot{y} + y\dot{x}$, $\therefore$ the fluxion of $xyz = xy\dot{z} + xz\dot{y} + zy\dot{x}$.

RULE. Hence the fluxion of the product of any number of flowing quantities is obtained by taking the sum of the products, which arise from multiplying to-gether the fluxion of each quantity, and the product of all the others.

(17.) To find the fluxion of any power of a simple quantity x^n. Let $x - x'$, x, and $x + x'$, be the preceding, present, and succeeding values of x. Then the corres-ponding values of x^n are,

$$(x - x')^n, \text{ or } x^n - nx^{n-1}x' + n \cdot \frac{n-1}{2}x^{n-2}x'^2 - \&c.$$

$$x^n, \text{ or } x^n,$$

$$\text{and } (x + x')^n, \text{ or } x^n + nx^{n-1}x' + n \cdot \frac{n-1}{2}x^{n-2}x'^2 + \&c.$$

$\therefore$ the preceding increment $= nx^{n-1}x' - n \cdot \dfrac{n-1}{2} x^{n-2}x'^{2} + \&c.$

the succeeding . . . . $= nx^{n-1}x' + n \cdot \dfrac{n-1}{2} x^{n-2}x'^{2} - \&c.$

and the limit $= nx^{n-1}x'$, or the fluxion $= nx^{n-1}\dot{x}.$

In this case, n may be either positive or negative, a whole number or a fraction.

(18.) To find the fluxion of any power of a compound quantity, as $(a^{m}+x^{m})^{\frac{p}{q}}.$

Let $(a^{m}+x^{m})^{\frac{p}{q}} = y$, $\therefore a^{m}+x^{m} = y^{\frac{q}{p}}$,

$\therefore mx^{m-1}\dot{x} = \dfrac{q}{p} \times y^{\frac{q}{p}-1}\dot{y}$,

$\therefore \dot{y} = \dfrac{p}{q} \times \dfrac{mx^{m-1}\dot{x}}{y^{\frac{q}{p}-1}} = \dfrac{p}{q} \times \dfrac{mx^{m-1}\dot{x}}{(a^{m}+x^{m})^{\frac{p}{q} \times \frac{q}{p}-1}}$

$\qquad = \dfrac{p}{q} \times \dfrac{mx^{m-1}\dot{x}}{(a^{m}+x^{m})^{1-\frac{p}{q}}} = \dfrac{p}{q} \times mx^{m-1}\dot{x} \times (a^{m}+x^{m})^{\frac{p}{q}-1};$

that is, the fluxion of $(a^{m}+x^{m})^{\frac{p}{q}} = \dfrac{p}{q} \times mx^{m-1}\dot{x} \times$

$(a^{m}+x^{m})^{\frac{p}{q}-1}.$

In cases of this kind $a^{m}+x^{m}$ is called the root, and the fluxion of this root is $mx^{m-1}\dot{x}.$

RULE. Hence the fluxion of any power of a flowing quantity, whether simple or compound, is found by multiplying together the index, the next inferior power, and the fluxion of the root.

(19.) To find the fluxion of a fraction, as $\dfrac{x}{y}.$

Let $\dfrac{x}{y} = z$, $\therefore x = yz$, $\therefore \dot{x} = y\dot{z} + z\dot{y}$, $\therefore y\dot{z} = \dot{x} - z\dot{y} =$

$\dot{x} - \dfrac{x\dot{y}}{y} = \dfrac{y\dot{x} - x\dot{y}}{y}$, $\therefore \dot{z} = \dfrac{y\dot{x} - x\dot{y}}{y^{2}}.$ But $\dot{z} =$ the fluxion of

$\dfrac{x}{y}$, $\therefore$ the fluxion of $\dfrac{x}{y} = \dfrac{y\dot{x} - x\dot{y}}{y^2}$.

Hence the RULE. From the fluxion of the numerator, multiplied into the denominator, subtract the fluxion of the denominator multiplied into the numerator, and divide by the square of the denominator.

(20.) EXAMPLES of the different RULES.

Ex. 1. The fluxion of $x^8 = 8x^7\dot{x}$.

Ex. 2. The fluxion of $x^9 = 9x^8\dot{x}$.

Ex. 3. The fluxion of $(a+x)^6 = 6\dot{x}(a+x)^5$.

Ex. 4. The fluxion of $(2ax - x^2)^2 = (4a\dot{x} - 4x\dot{x})$ $(2ax - x^2)$.

Ex. 5. The fluxion of $(a^2 - x^2)^{\frac{4}{5}} = \dfrac{4}{5} \times -2x\dot{x}(a^2 - x^2)^{-\frac{1}{5}}$

$= \dfrac{-8x\dot{x}}{5(a^2 - x^2)^{\frac{1}{5}}}$.

Ex. 6. The fluxion of $(ax^2 + bx^3 - cx^4)^{\frac{1}{2}} = (3ax\dot{x} + \dfrac{9bx^2\dot{x}}{2} - 6cx^3\dot{x})\ (ax^2 + bx^3 - cx^4)^{\frac{1}{2}}$.

Ex. 7. The fluxion of $x^3y^4 = 3x^2\dot{x} \times y^4 + 4y^3\dot{y} \times x^3$
$= 3y^4x^2\dot{x} + 4x^3 \times y^3\dot{y} = x^2y^3\ (3y\dot{x} + 4xy\dot{y})$.

Ex. 8. The fluxion of $x^2y^2z^{\frac{5}{3}} = 2y^2z^{\frac{5}{3}}x\dot{x} + 3x^2z^{\frac{5}{3}}y^2\dot{y} + \dfrac{5}{3} \times$

$x^2y^3z^{\frac{2}{3}}\dot{z}$.

Ex. 9. The fluxion of $x(a^2 + x^2)^{\frac{1}{2}} = \dot{x}(a^2 + x^2)^{\frac{1}{2}} + 3x\dot{x}$ $(a^2 + x^2)^{\frac{1}{2}}\ x = (a^2\dot{x} + x^2\dot{x} + 3x^2\dot{x})\ (a^2 + x^2)^{\frac{1}{2}} = (a^2\dot{x} + 4x^2\dot{x})$ $(a^2 + x^2)^{\frac{1}{2}} = (a^2 + 4x^2)\ (a^2 + x^2)^{\frac{1}{2}}\dot{x}$.

Ex. 10. The fluxion of $(x^2 + y^2)^{\frac{1}{2}}\ (x^2 - y^2)^{\frac{4}{3}} =$ $(3x\dot{x} + 3y\dot{y})\ (x^2 + y^2)^{\frac{1}{2}}\ (x^2 - y^2)^{\frac{4}{3}} + \dfrac{8x\dot{x} - 8y\dot{y}}{3}\ (x^2 - y^2)^{\frac{1}{3}}$ $(x^2 + y^2)^{\frac{1}{2}}$.

Ex. 11. The fluxion of $(a^2+x^2)^{\frac{1}{2}}$ $(b^2-y^2)^{\frac{2}{3}} = x\dot{x}$ $(a^2+x^2)^{-\frac{1}{2}}$ $(b^2-y^2)^{\frac{2}{3}} - 3y\dot{y}$ $(a^2+x^2)^{\frac{1}{2}}$ $(b^2-y^2)^{\frac{1}{2}} = \dfrac{ax\dot{x}\ (b^2-y^2)^{\frac{2}{3}}}{(a^2+x^2)^{\frac{1}{2}}} - 3y\dot{y}$ $(a^2+x^2)^{\frac{1}{2}}$ $(b^2-y^2)^{\frac{1}{2}}$.

Ex. 12. The fluxion of $\dfrac{x^2}{y^2} = \dfrac{2x\dot{x}\times y^2 - 2y\dot{y}\times x^2}{y^4} = \dfrac{2y^2x\dot{x}-2x^2y\dot{y}}{y^4}$.

Ex. 13. The fluxion of $\dfrac{1}{x^6} = \dfrac{0-6x^5\dot{x}}{x^{12}} = \dfrac{-6x^5\dot{x}}{x^{12}} = \dfrac{-6\dot{x}}{x^7}$.

Ex. 14. The fluxion of $\dfrac{a}{x^n} = \dfrac{0-nx^{n-1}\dot{x}\times a}{x^{2n}} = \dfrac{-na\dot{x}}{x^{n+1}}$.

Ex. 15. The flux. of $\dfrac{x-y}{x^3} = \dfrac{(\dot{x}-\dot{y})\ x^3 - 3x^2\dot{x}\ (x-y)}{x^6}$ $= \dfrac{x^3\dot{x}-x^3\dot{y}-3x^3\dot{x}+3yx^2\dot{x}}{x^6} = \dfrac{3yx^2\dot{x}-2x^3\dot{x}-x^3\dot{y}}{x^6}$.

Ex. 16. The fluxion of $\dfrac{x}{(a+x)^5} =$ flux. of $x\,(a+x)^{-5}$ $= \dot{x}\ (a+x)^{-5} - 5\dot{x}\ (a+x)^{-6}\ x = (a\dot{x} + x\dot{x} - 5x\dot{x})$ $(a+x)^{-6} = \dfrac{a\dot{x}-4x\dot{x}}{(a+x)^6}$.

Ex. 17. The fluxion of $\sqrt{\dfrac{a+x}{b+y}} = (a+x)^{\frac{1}{2}}\ (b+y)^{-\frac{1}{2}} = \frac{1}{2}\dot{x}\ (a+x)^{-\frac{1}{2}}\ (b+y)^{-\frac{1}{2}} - \frac{1}{2}\dot{y}\ (a+x)^{\frac{1}{2}}\ (b+y)^{-\frac{3}{2}} = \dfrac{\dot{x}}{2\ (a+x)^{\frac{1}{2}}\ (b+y)^{\frac{1}{2}}} - \dfrac{\dot{y}(a+x)^{\frac{1}{2}}}{2(b+y)^{\frac{3}{2}}}$.

Ex. 18. The fluxion of $\dfrac{\sqrt{a^3+x^3}}{(a^2+y^2)^{\frac{1}{3}}} =$ fluxion of $(a^3+x^3)^{\frac{1}{2}}$ $(a^2+y^2)^{-\frac{1}{3}} = \dfrac{3x^2\dot{x}}{2}\ (a^3+x^3)^{-\frac{1}{2}}\ (a^2+y^2)^{-\frac{1}{3}} - \dfrac{2y\dot{y}}{3}\ (a^2+y^2)^{-\frac{4}{3}}$

$$(a^3+x^2)^{\frac{1}{2}} = \frac{3x^2\dot{x}}{2\,(a^3+x^3)^{\frac{1}{2}}\,(a^2+y^2)^{\frac{1}{3}}} - \frac{2y\dot{y}\,(a^3+x^3)^{\frac{1}{2}}}{3\,(a^2+y^2)^{\frac{4}{3}}}.$$

Ex. 19. The fluxion of $\dfrac{x-1}{\sqrt{2x-x^2}}$, or of $(x-1)$

$$(2x-x^2)^{-\frac{1}{2}} = \frac{\dot{x}\,(2x-x^2)}{(2x-x^2)^{\frac{3}{2}}} + \frac{(x\dot{x}-\dot{x})\,(x-1)}{(2x-x^2)^{\frac{3}{2}}} = \frac{\dot{x}}{(2x-x^2)^{\frac{3}{2}}}.$$

Ex. 20. The fluxion of $-\dfrac{x+1}{\sqrt{2x+x^2}} = \dfrac{\dot{x}}{(2x+x^2)^{\frac{3}{2}}}$.

Ex. 21. The fluxion of $\dfrac{-\sqrt{a^2-x^2}}{a^2x} = \dfrac{\dot{x}}{x^2\times\sqrt{a^2-x^3}}$.
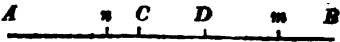
Ex. 22. The fluxion of $\dfrac{-(a^2+x^2)^{\frac{3}{2}}}{3a^2x^3} = \dfrac{\dot{x}\times\sqrt{a^2+x^2}}{x^4}$.

Ex. 23. The flux. of $\dfrac{y^2}{\sqrt{y^2+b^2}} = $ flux. of $y^2\,(y^2+b^2)^{-\frac{1}{2}}$

$$= 2y\dot{y}\,(y^2+b^2)^{-\frac{1}{2}} \frac{-y\dot{y}\times y^2}{(y^2+b^2)^{\frac{3}{2}}} = \frac{2y^3\dot{y}+2b^2y\dot{y}-y^3\dot{y}}{(y^2+b^2)^{\frac{3}{2}}} =$$

$$\frac{y^3\dot{y}+2b^2y\dot{y}}{(y^2+b^2)^{\frac{3}{2}}}.$$

CHAP. II.

ON THE MAXIMA AND MINIMA OF QUANTITIES.

(21.) THE fluxion of a quantity, when it is a maximum or minimum, $= 0$.

Let two points, m and n, begin to move on the straight line AB at the same time; m from the point C with a given uniform velocity, and n from A with a velocity less than that of m at the commencement of the motion, but which is continually increasing. Let D be the point, in which the velocity of n equals that of m; it is evident that the distance nm is perpetually increasing till n arrives at D, and then n begins to overtake m, or nm decreases. At D therefore the distance between n and m neither increases nor decreases; and consequently its fluxion $= 0$. But in this case nm is a maximum; hence the fluxion of a quantity, when it is a maximum, $= 0$.

Next, let the velocity of n at first, be greater than that of m at first, but perpetually decrease; $n\,m$ continually decreases till the velocity of n is equal to that of m, and afterwards it increases. Let D be the point, in which the velocity of n is equal to that of m; here then $n\,m$ is neither increasing nor decreasing; therefore its fluxion is nothing; but here nm is a minimum; consequently the fluxion of a quantity, which is a minimum, $= 0$.

(22.) If a quantity be a maximum or a minimum, any power or root of that quantity is a maximum or a mini-

mum ; for the increase or decrease of the power or root
will depend upon the increase or decrease of the original
quantity.

In the same manner any constant multiple, or part of
the original quantity, is at the same time a maximum or
minimum.

(23.) EXAMPLES.

Ex. 1. To find the greatest ordinate in a given circle.

Let $AM=x$; $AB=2a$; $BM=$
$2a-x$; then $MP^2=2ax-x^2$ a
maximum ; $\therefore 2a\dot{x} - 2x\dot{x} = 0$,
and $x=a$; that is, if C be the
centre, and CD be drawn per-
pendicular to AB, it is the or-
dinate required.
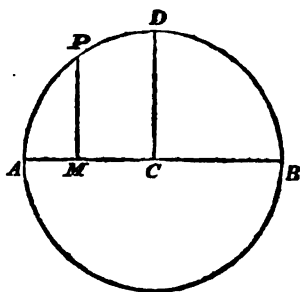
Ex. 2. To divide a given line AB into two parts x
and y, so that the rectangle xy may be a maximum.

Let $AC=x$; $CB=y$; $AB=a$;
then $x + y = a$, and $\dot{x} + \dot{y} = 0$;
$\therefore \dot{x} = -\dot{y}$; also xy is a maxi-
mum ; $\therefore x\dot{y} + y\dot{x} = 0$, or substituting $-\dot{y}$ for $\dot{x}, -x\dot{x} + y\dot{x}$
$=0$; hence $x=y$, or AB must be bisected in C.

Cor. Hence to divide a quantity a into three parts
x, y, and z, so that their continued product may be a
maximum, the parts must be equal. For if x be assumed
constant, the product yz, and therefore xyz is a maximum,
when $y=z$; if y be assumed constant, the product is
a maximum, when $x=z$; if z be constant, x must equal y.
And in the same manner, into whatever number of parts
a given line is divided, they are all equal, when their
continued product is a maximum.

Ex. 3. To divide a given line AB into two parts x and y, so that $x^m \times y^n$ may be a maximum.

Let $AB = a$; $AC = x$; $CB = y$. Then $x + y = a$, and $\dot{x} = -\dot{y}$.

Also $x^m \times y^n$ is a maximum; hence $m x^{m-1} \dot{x} \times y^n + n y^{n-1} \dot{y} \times x^m = 0$. Divide both sides by $x^{m-1} y^{n-1}$, then $m y \dot{x} + n x \dot{y} = 0$; for $\dot{x}$ write $-\dot{y}$; $\therefore n x \dot{y} = m y \dot{y}$, and $x : y :: m : n$.

Cor. If it be required to divide the given line into three parts x, y, and z, so that $x^m \times y^n \times z^p$ may be a maximum; it will follow by the reasoning in the last Cor. that $x : y :: m : n$, or $y = \dfrac{nx}{m}$; and $x : z :: m : p$; $\therefore z = \dfrac{px}{m}$; hence, since $x + y + z = a$, we have $x + \dfrac{nx}{m} + \dfrac{px}{m} = a$, and $x = \dfrac{ma}{m + n + p}$. We may proceed in the same manner, whatever be the number of unknown quantities.

Ex. 4. To divide a given line AB into two parts, x and y, so that $\dfrac{x}{y} + \dfrac{y}{x}$ may be a minimum.

As before, $x + y = a$, and $\dot{x} = -\dot{y}$; also $\dfrac{x}{y} + \dfrac{y}{x}$ is a minimum; $\therefore \dfrac{\dot{x} y - \dot{y} x}{y^2} + \dfrac{\dot{y} x - \dot{x} y}{x^2} = 0$, or $-\dfrac{y \dot{y} + x \dot{y}}{y^2} + \dfrac{x \dot{y} + y \dot{y}}{x^2} = 0$, by writing $-\dot{y}$ for $\dot{x}$; that is, $\dfrac{x \dot{y} + y \dot{y}}{y^2} = \dfrac{y \dot{y} + x \dot{y}}{x^2}$; $\therefore \dfrac{1}{y^2} = \dfrac{1}{x^2}$ and $x = y$, or the line must be bisected.

Ex. 5. To find the fraction, which shall exceed its cube by the greatest quantity possible.

Let the required fraction $= x$; then $x - x^3$ is a

maximum; $\therefore \dot{x} - 3x^2\dot{x} = 0$, and $x = \dfrac{1}{\sqrt{3}} = \dfrac{1}{3} \times \sqrt{3}$.

Ex. 6. To inscribe the greatest rectangle in a given triangle, ABC.

Draw AD perpendicular to BC, take $AD = a$, $BC = b$, $AI = x$, $EF = y$; then, by similar triangles,

$$a : b :: x : y = \frac{bx}{a};$$
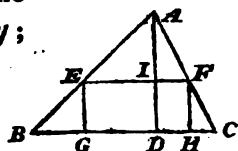
$\therefore$ the rectangle $EGHF = \dfrac{bx}{a} \times \overline{a - x}$ is a maximum;

$\therefore ax - x^2$ is a maximum; $\therefore x = \dfrac{a}{2}$.

Ex. 7. To inscribe the greatest rectangle in a given ellipse ABN.

Let $CM = x$; PM the ordinate $= y$; then, if AC the semi-major axis $= a$, and BC the semi-minor $= b$, the rectangle $PEFD = 2$

$\times \dfrac{b}{a} \times \sqrt{\overline{a^2 - x^2}} \times 2x$;
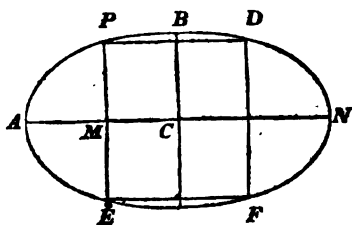
$\therefore x \times \sqrt{\overline{a^2 - x^2}}$, or its square $a^2x^2 - x^4$, is a maximum;

$\therefore$ the fluxion $= 0$, and $x = \dfrac{a}{\sqrt{2}} = \dfrac{1}{2} a\sqrt{2}$.

Ex. 8. To inscribe the greatest rectangle in a given parabola ABC.

Draw the axis AD; let $AI = x$, $IF = y$, $AD = b$; latus rectum $= c$; then $FI = \sqrt{cx}$; $\therefore$ the rectangle $EH = 2\sqrt{cx}\,(b - x)$; $\therefore bx^{\frac{1}{2}} - x^{\frac{3}{2}}$ is a maximum; its fluxion $= 0$, and $x = \dfrac{b}{3}$.
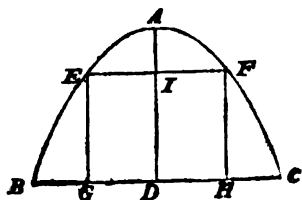
Ex. 9. Given the radius of a circle to determine the arc, when the rectangle under the sine and cosine is a maximum.

Let the radius $= r$, and the cosine $= x$; then the sine $= \sqrt{r^2 - x^2}$; $\therefore$ by the problem $x \times \sqrt{r^2 - x^2}$ is a maximum; hence the fluxion of $x^2 \times \overline{r^2 - x^2} = 0$, and $x = \dfrac{r}{\sqrt{2}} = \dfrac{1}{2} r \times \sqrt{2}$.

Ex. 10. To determine in an ellipse, at what point the angle contained between the tangent and distance, is a minimum.

Let S and H be the foci of the ellipse, P the point required, PY a tangent, SY perpendicular to PY, and SPY the least angle. Its sine is a minimum. Let $x =$ the sine to radius 1; then
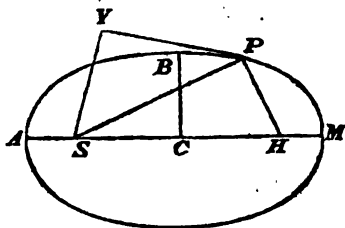
$1 : x :: SP : SY$; $\therefore x = \dfrac{SY}{SP} = \dfrac{1}{SP} \times BC \times$

$\sqrt{\dfrac{SP}{HP}} \propto \dfrac{1}{\sqrt{SP \times HP}}$ a minimum; $\therefore SP \times HP$ is

a maximum; and since $SP + HP = AM$, in this case $SP = HP$; and the point P is at the extremity of the minor axis.

Ex. 11. Of all right-angled plane triangles having the same given hypothenuse, to find that whose area is greatest.

Let $AB = a$, $BC = x$; then $AC = \sqrt{a^2 - x^2}$. Now the area of the triangle $ABC = \dfrac{BC \times AC}{2}$; $\therefore \dfrac{x \times \sqrt{a^2 - x^2}}{2}$, or $x^2 (a^2 - x^2)$ is a maximum; $\therefore$ the fluxion of $a^2 x^2 - x^4$, or $2 a^2 x \dot{x} - 4 x^3 \dot{x}$
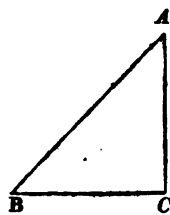
$= 0$; hence $a^2 = 2x^2$, and $x = \dfrac{a}{\sqrt{2}} = \dfrac{1}{2} a \sqrt{2} = BC$
$= AC$.

Ex. 12. Of all right-angled triangles having the same area, to find that in which the sum of the sides BC, CA is a minimum.

Let the area $= a$, $BC = x$; then, since $BC \times AC$
$= 2a$, $AC = \dfrac{2a}{x}$; $\therefore x + \dfrac{2a}{x}$ is a minimum; hence

$\dot{x} - \dfrac{2a\dot{x}}{x^2} = 0$, and $x = \sqrt{2a}$.

Ex. 13. To find the value of x and y in the equation $(x^2 + y^2)^2 = a^2 x^2$, when y is a maximum.

Extract the square root; then $x^2 + y^2 = ax$; $\therefore 2x\dot{x}$
$+ 2y\dot{y} = a\dot{x}$; but $\dot{y}$, and consequently $2y\dot{y} = 0$, when

y is a maximum; $\therefore 2x\dot{x} = a\dot{x}$, and $x = \dfrac{a}{2}$. To find y,

we have $\dfrac{a^2}{4} + y^2 = \dfrac{a^2}{2}$; $\therefore y^2 = \dfrac{a^2}{4}$, and $y = \dfrac{a}{2}$.

(24.) To determine when the equation $x^3 - 9x^2 + 24x - 16 = 0$ becomes a maximum or minimum.

Assume the fluxion $= 0$; then $3x^2\dot{x} - 18x\dot{x} + 24\dot{x}$
$= 0$; or $3\dot{x} (x^2 - 6x + 8) = 0$.

Now the roots of this quadratic equation are 2 and 4;
$\therefore 3\dot{x} (x - 2) (x - 4) = 0$.

To ascertain which of these roots gives a maximum, and which a minimum; find whether the value of the fluxion just before it $= 0$, be positive or negative. If it be positive, the quantity is increasing, and the next root gives a maximum; if negative, it is decreasing, and the next root gives a minimum.

In this instance, if $\dot{x}$ be assumed positive, and x less than 2, the value of $3\dot{x} (x - 2) (x - 4)$ is positive; $\therefore$ this root gives $x^3 - 9x^2 + 24x - 16$ a maximum.

If $\dot{x}$ be assumed greater than 2, but less than 4, $3\dot{x}$ $(x-2)$ $(x-4)$ is negative; ∴ this root gives the original equation a minimum.

(25.) The meaning of the assertion, that if $x = 2$ it gives the equation a maximum, and if it $= 4$ a minimum, is, that $x^3 - 9x^2 + 24x - 16$ increases till $x = 2$, and then decreases till $x = 4$; not that it is the greatest possible when $x = 2$, nor the least possible when $x = 4$. For if quantities less than 2 be successively substituted for x, as

$$
\left.\begin{array}{l} 1 \\ 0 \\ -1 \text{ \&c.} \end{array}\right\} \text{- - - - - - - the results are} \left\{\begin{array}{l} 1-9+24-16, \text{ or } 0, \\ -16, \\ -1-9-24-16, \text{ or } -50 \text{ \&c.} \end{array}\right.
$$

that is, it will go on decreasing, *sine limite*.

And if quantities greater than 4 be substituted successively for x, as

$$
\left.\begin{array}{l} 5 \\ 6 \\ 7 \text{ \&c.} \end{array}\right\} \text{- - - - - - - the results are} \left\{\begin{array}{l} 125-225+120-16, \text{ or } 4, \\ 216-324+144-16, \text{ or } 20, \\ 343-441+168-16, \text{ or } 54; \end{array}\right.
$$

that is, it will go on increasing, *sine limite*.

(26.) In this case we have supposed x to increase, and therefore that $\dot{x}$ is positive. · If x be a decreasing quantity, its fluxion is negative. Suppose x to decrease till it becomes equal to 4; here $3\dot{x}$ $(x-2)$ $(x-4)$ is negative, while x is greater than 4; therefore, when $x = 4$, the original quantity $x^3 - 9x^2 + 24x - 16$ is a minimum. If x be assumed greater than 2, and less than 4, then $3\dot{x}$ $(x-2)$ $(x-4)$ is positive; therefore the root 2 gives $x^3 - 9x^2 + 24x - 16$ a maximum. These results are exactly the same with those obtained by the first method.

(27.) When two or an even number of the roots of the resulting equation are equal, they shew neither a maximum nor a minimum.

It follows from the preceding articles, that when the fluxion of the given quantity is of the same denomination with regard to positive and negative, before and after it becomes equal to nothing, it does not indicate either a maximum or minimum. Now this occurs, when two roots of the fluxional equation are equal. For, let the given quantity be $3x^4 - 32x^3 + 120x^2 - 192x$; of which the fluxion is $12x^3\dot{x} - 96x^2\dot{x} + 240x\dot{x} - 192\dot{x}$;

$$\text{or, } 12\dot{x} \ (x^3 - 8x^2 + 20x - 16) \ ;$$
$$\text{or, } 12\dot{x} \ (x - 2) \ (x - 2) \ (x - 4).$$

Let $\dot{x}$ be positive; then before $x = 2$ this fluxion is negative; and if x be greater than 2, and less than 4, it is still negative; therefore the root 2 does not give a minimum. But as the fluxion changes from $-$ to $+$, while x increases from a quantity less than 4, to a quantity greater than 4, this root 4 gives $3x^4 - 32x^3 + 120x^2 - 192x$ a minimum; and it then begins to increase.

In the same manner, if the fluxional equation has 4 equal roots, as $\dot{x} \ (\overline{x - a} \times \overline{x - a} \times \overline{x - a} \times \overline{x - a} \times \overline{x - 2a},)$ or any even number, the fluxion is of the same denomination with respect to $+$ and $-$, both before and after x becomes equal to a; and therefore the equal roots neither indicate a maximum nor a minimum.

(28.) The number of maxima or minima which a flowing quantity admits, is equal to the number of unequal roots in the fluxional equation.

Let $3x^4 - 28ax^3 + 84a^2 x^2 - 96a^3 x + 48b^4 = 0$ be an equation, in which it is required to determine the different values of x, when the expression becomes a maximum or minimum. Put the fluxion $= 0$;

$$\therefore \ 12x^3\dot{x} - 84ax^2\dot{x} + 168a^2 x\dot{x} - 96a^3\dot{x} = 0;$$
$$\text{or, } 12\dot{x} \ (x^3 - 7ax^2 + 14a^2 x - 8a^3) = 0;$$
$$\text{or, } 12\dot{x} \ (x - a) \ (x - 2a) \ (x - 4a).$$

If x be assumed less than a, the result is —, or the root a indicates a minimum; if x be greater than a, but less than $2a$, the result is + ; and the root $2a$ denotes a maximum, &c. ; therefore when all the roots are unequal, the proposition is true.

And if the fluxional equation have an odd number of equal roots, as $\dot{x}\,(x-a)\,(x-a)\,(x-a)\,(x-2a)$ when x is less than a, the result is + ; when greater than a, but less than $2a$, it is — ; therefore one root a gives a maximum, and $2a$ a minimum; the product of $(x-a)$ $(x-a)$ determines nothing; hence universally, there are as many maxima and minima, as unequal roots, in the given equation.

When all the roots are impossible in the fluxional equation, as no possible value of x can give a result $= 0$, the quantity must either increase or decrease perpetually, and therefore cannot admit a maximum or minimum.

(29.) Every quantity which admits a maximum or minimum is of a compound nature; one part of it must increase, while another decreases, and according to the increase or decrease it approaches a maximum or a minimum. Thus, in Ex. 1. Art. 23, we have two quantities, $2ax$ and $-x^2$; if x increases, $2ax$ also increases; but x^2 increases at the same time; therefore the expression $2ax - x^2$ partly increases, and partly diminishes; this quantity, then, is in a state to admit a maximum or a minimum.

CHAP. III.

THE METHOD OF DRAWING TANGENTS TO CURVES.

(30.) IF a straight line as BC move on the line AC in a direction always parallel to itself, and AC and CB increase uniformly, the locus of the point B will be a straight line; and the motion of B in that straight line will be uniform.

Let BC come into the position DE. Then, since AC and CB begin their motion together, and have an uniform increase, the ratio of AC : CB is constant; that is, AC : CB :: AE : ED, or ACB and AED are similar triangles; $\therefore ABD$ is a straight line. Also the motion in that line is uniform; for, since CB and ED are parallel, AC : CE :: AB : BD, and alternately AC : AB :: CE : BD; but AC is to AB in a constant ratio; $\therefore CE$: BD in a constant ratio; and the motion in the direction AE is uniform; $\therefore$ the motion in the direction AD is uniform.
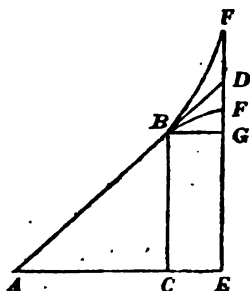
COR. 1. If the motion in the direction AC be uniform, but that in the direction CB not uniform, the point B will trace out a curve. The same construction remaining, let CB increase with an accelerated velocity; then BG being drawn parallel to CE, BG and GD would represent the uniform contemporaneous increments of AC, and CB; but if CB increase with an

accelerated velocity whilst the velocity of C is uniform, BG and some line GF greater than GD will represent the corresponding increments; in this case, a curve BF is described convex to the line AE. By the same reasoning, if the increment of CB is perpetually retarded, whilst that of AC remains uniform, the point B will trace out a curve, which is concave to the line AE.

Cor. 2. If AC increase with an uniform velocity, but the increase of CB is not uniform, the curve BF is not described with an uniform motion.

(31.) To draw a tangent to any algebraic curve. Let AM represent the abscissa, and MP the ordinate of an algebraic curve convex to the axis AF. Take Mc
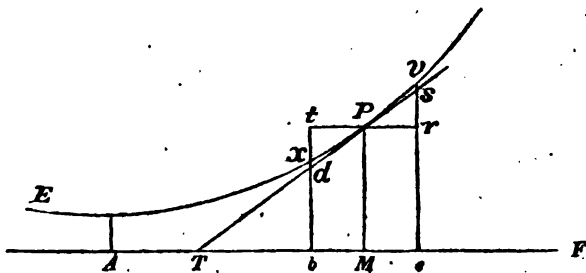
and Mb on each side of PM, and equal to each other; and draw bt and cv parallel to PM, meeting the curve in x and v, and a line tPr parallel to AF in t and r. Let Mb, and Mc, or Pt and Pr represent the uniform increase of the abscissa AM in a given time. Then since EPV is convex to the axis, MP increases with an accelerated velocity; $\therefore$ the fluxion of AM : the fluxion of MP :: Pr : a quantity less than rv. Take rs equal to this quantity; join Ps, and produce it both ways; this line is a tangent, that is, every part of it falls below the curve. For since by equal triangles Ptd, Prs, $rs = td$; $\therefore$ td is the fluxion of the ordinate at P. But the fluxion of AM : the fluxion of MP :: Pr or Pt :

a quantity greater·than the preceding increments tx; ∴ td is greater than tx, and d is below the curve.

Next, suppose the curve to be concave to the axis AF. The same construction remaining, since the increase of
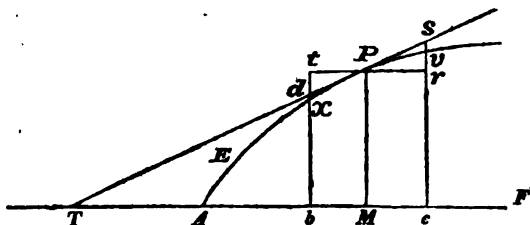
MP is in this case retarded, the fluxion of AM : the fluxion of MP :: Pr : a quantity greater than rv; take rs equal to this quantity; join sP, and produce it; this line is a tangent at P, or falls wholly above the curve. For, by equal and similar triangles, Prs, Ptd, $td=rs$, or = the fluxion of the ordinate at P.

But the fluxion of AM : the fluxion of MP :: Pt or Pr : a quantity less than tx; ∴ td is less than tx, or d is above the curve.

In both these cases, if $AM=x$, $PM=y$; $Mc=\dot{x}$, $rs=\dot{y}$, we have rs : rP or Mc :: PM : MT by similar triangles;

or $\dot{y}$: $\dot{x}$ :: y : the subtangent $MT=\dfrac{y\dot{x}}{\dot{y}}$.

Hence in any algebraic curve, to which it is required to draw a tangent at any point P, find from the equation to the curve the value of $\dfrac{y\dot{x}}{\dot{y}}$; take MT equal to this expression, join TP, and produce it. TP is the tangent required.

EXAMPLES.

(32.) Ex. 1. To draw a tangent to the common parabola.

Let AP be the parabola, AM the axis, and P the point at which the tangent is to be drawn. Take $AM = x$, $PM = y$, the principal latus rectum $= a$; then $y^2 = ax$;
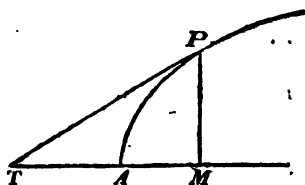
$$\therefore 2y\dot{y} = a\dot{x}; \quad \therefore \frac{\dot{x}}{\dot{y}} = \frac{2y}{a}, \text{ and } \frac{y\dot{x}}{\dot{y}} = \frac{2y^2}{a} = \frac{2ax}{a} = 2x;$$

that is, MT the subtangent = twice the abscissa MA. Hence, to draw a tangent at P, let fall the ordinate PM, and in AM produced, take $MT = 2MA$, and join TP. TP is the tangent.

Ex. 2. In general, to draw a tangent to any parabola of which the equation is $a^{n-1}x = y^n$.

Here $a^{n-1}\dot{x} = ny^{n-1}\dot{y}$; $\therefore \frac{\dot{x}}{\dot{y}} = \frac{ny^{n-1}}{a^{n-1}}$; $\therefore MT$ or $\frac{y\dot{x}}{\dot{y}} =$

$$\frac{ny^n}{a^{n-1}} = \frac{na^{n-1}x}{a^{n-1}} = nx \text{ the subtangent.}$$

Cor. If $n = 2$ it becomes the common parabola, and $MT = 2x$ as before.

Ex. 3. To draw a tangent to the circle at a given point P.

Let fall the ordinate PM on the diameter AD. Let $AM = x$; $PM = y$; $AD = 2a$; then $y^2 = 2ax - x^2$; $\therefore 2y\dot{y} = 2a\dot{x} - 2x\dot{x}$; $\therefore \frac{\dot{x}}{\dot{y}} = \frac{y}{a-x}$, $\therefore \frac{y\dot{x}}{\dot{y}}$

or $MT = \frac{y^2}{a-x} = \frac{2ax - x^2}{a-x}$.
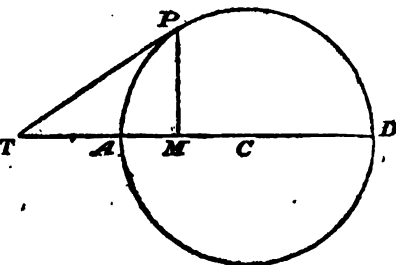
Take $MT = $ to this quantity, and join TP; TP is the tangent. Since $CM = a - x$, and $PM = \sqrt{2ax - x^2}$,

$\therefore CM : MP :: MP : MT = \dfrac{2ax - x^2}{a - x}$, and consequently,

if CP be joined, CPT is a right angle.

Ex. 4. To draw a tangent to the ellipse ABD at any point P.

Let $AC = a$, $CB = b$, where AC and CB are the semi-axes;

$AM = x, PM = y,$
then, by the property of
the ellipse, $AM \times MD$
$: MP^2 :: AC^2 : BC^2$,
or $x\,(2a - x) : y^2 :: a^2 :$

b^2; $\therefore y^2 = \dfrac{b^2}{a^2}(2ax - x^2)$;

$\therefore 2y\dot{y} = \dfrac{b^2}{a^2}(2a\dot{x} - 2x\dot{x}) = \dfrac{b^2}{a^2}(a - x)\,2\dot{x}, \; \therefore \dfrac{\dot{x}}{\dot{y}} = \dfrac{y}{\dfrac{b^2}{a^2}(a - x)}$;

$\therefore MT \text{ or } \dfrac{y\dot{x}}{\dot{y}} = \dfrac{y^2}{\dfrac{b^2}{a^2}(a - x)} = \dfrac{\dfrac{b^2}{a^2}(2ax - x^2)}{\dfrac{b^2}{a^2}(a - x)} = \dfrac{2ax - x^2}{a - x}.$
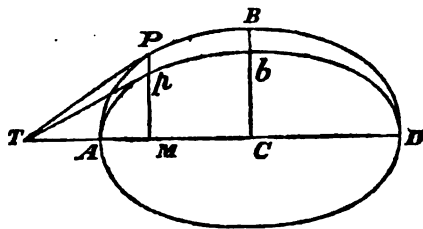
Cor. 1. Since the value of MT is expressed in terms of a and x, the subtangent will continue the same, if a and x are the same, whatever be the minor axis of the ellipse; therefore if any other ellipse ApD be described on the same major axis, and cut MP in p, and Tp be joined, Tp is also a tangent to this ellipse at the point p. And if a circle, described upon the major axis, as a diameter, cut MP produced in R, and TR be joined, TR is a tangent to the circle.

Cor. 2. If x be greater than a, and less than $2a$, $\dfrac{2ax - x^2}{a - x}$ is negative; or the sub-tangent lies the contrary way.

Ex. 5. To draw a tangent to the hyperbola AP, whose major axis is AD, and minor $2BC$.

Let $AC = a$; $CB = b$; $AM = x$; $MP = y$.
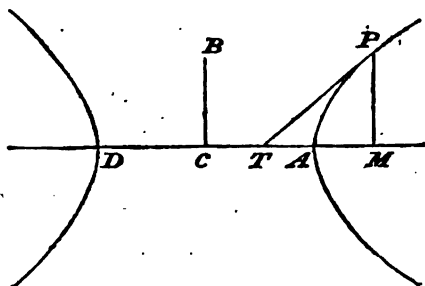
By the nature of the hyperbola,

$$AM \times MD : MP^2 :: AC^2 : CB^2 ;$$

$$\text{or, } x\,(2a+x) : y^2 :: a^2 : b^2 ;$$

$$\therefore y^2 = \frac{b^2}{a^2}(2ax + x^2) ;$$

$$\therefore 2y\dot{y} = \frac{b^2}{a^2}(2a\dot{x} + 2x\dot{x}) = \frac{b^2}{a^2}(a+x)\,2\dot{x} ;$$

$$\therefore \frac{\dot{x}}{\dot{y}} = \frac{y}{\dfrac{b^2}{a^2}(a+x)} ; \quad \therefore MT \text{ or } \frac{y\dot{x}}{\dot{y}} = \frac{y^2}{\dfrac{b^2}{a^2}(a+x)} =$$

$$\frac{\dfrac{b^2}{a^2}(2ax + x^2)}{\dfrac{b^2}{a^2}(a+x)} = \frac{2ax + x^2}{a+x}.$$

(33.) DEF. If PN be drawn perpendicular to PT, meeting the axis in N, PN is called the normal, and MN the sub-normal.
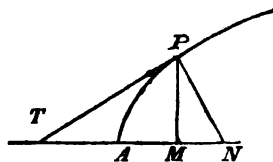
To determine their values, we have $TM : MP :: MP : MN$; or $\dfrac{y\dot{x}}{\dot{y}} : y :: y : MN$ the sub-normal $= \dfrac{y\dot{y}}{\dot{x}}$.

Also, $NP^2 = NM^2 + MP^2 = \dfrac{y^2\dot{y}^2}{\dot{x}^2} + y^2 = \dfrac{y^2\,(\dot{x}^2 + \dot{y}^2)}{\dot{x}^2}$;

$\therefore NP$ the normal $= \dfrac{y \times \sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}}$.

(34.)   Examples.

Ex. 1. To find the value of the normal and sub-normal in the common parabola.

Here $y^2 = ax$; $\therefore$ $2y\dot{y} = a\dot{x}$, and $\dfrac{y\dot{y}}{\dot{x}} = \dfrac{a}{2}$ the sub-normal;

Also, $\dot{x} = \dfrac{2y\dot{y}}{a}$; $\therefore$ $y^2 \times \dfrac{\dot{x}^2 + \dot{y}^2}{\dot{x}^2} = y^2 \left(1 + \dfrac{a^2\dot{y}^2}{4y^2\dot{y}^2}\right) =$

$y^2 + \dfrac{a^2}{4}$; $\therefore$ the normal $PN = \sqrt{y^2 + \tfrac{1}{4}a^2}$.

Ex. 2. To find the value of the sub-normal in the ellipse and hyperbola.

In the ellipse, $y^2 = \dfrac{b^2}{a^2} (2ax - x^2)$;

$$\therefore y\dot{y} = \dfrac{b^2}{a^2} (a\dot{x} - x\dot{x});$$

and $\dfrac{y\dot{y}}{\dot{x}} = \dfrac{b^2}{a^2} (a - x) =$ the sub-normal.

In the hyperbola, it $= \dfrac{b^2}{a^2} (a + x)$.

Chap. IV.

TO DRAW ASYMPTOTES TO CURVES.

(35.) Def. An asymptote to a curve is a straight line cutting the axis at a finite distance, which continually approaches the curve, and arrives nearer than by any assignable difference, but never meets it, though indefinitely produced.
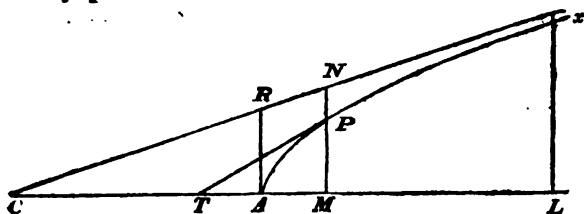
Let AP represent a curve, which admits an asymptote Cx; this line is conceived to be a tangent to the curve, or the limit to which the tangent approaches at an infinite distance. Take $AM = x$, $MP = y$; let MP produced meet the asymptote in N, and draw AR perpendicular to AM, and PT a tangent at P. From the nature of the curve proposed, find the value of $\dfrac{y\dot{x}}{\dot{y}} = MT$

the sub-tangent; hence AT may be found $= \dfrac{y\dot{x}}{\dot{y}} - x$.

Imagine x to become infinite, and T to move on to C; if AC be finite, the curve admits an asymptote. Next, find the ratio of $TM : MP$ when x is infinite; that is, Lx being supposed an ordinate at an infinite distance, the ratio of $CL : Lx$; then, by similar triangles, CLx, CAR, $CL : Lx :: CA : AR$; of which proportion the

three first terms are known, and therefore AR can be determined. Join CR, and produce it indefinitely. CR is the asymptote required.

EXAMPLES.

(36.) Ex. 1. To draw an asymptote to the common hyperbola. (See the preceding FIGURE.)

By Art. 32. Ex. 5, $MT = \dfrac{2ax + x^2}{a + x}$; $\therefore AT = \dfrac{2ax + x^2}{a + x} - x$

$= \dfrac{ax}{a + x}$. Let x become infinite; AT ultimately $= \dfrac{ax}{x}$

$= a = AC$. Now if x is infinite, TM, which then becomes $TL, = \dfrac{x^2}{x} = x$; and MP, or $\dfrac{b}{a} \times \sqrt{2ax + x^2}$, which

in that case $= Lx = \dfrac{bx}{a}$; $\therefore$ since $CL : Lx :: CA : AR$,

$$x : \frac{bx}{a} :: a : AR; \quad \therefore AR = b.$$

Hence, from A draw AR perpendicular to CA, and equal to b. Take the centre C; join CR, and produce it indefinitely. CRx is the asymptote.

Ex. 2. Let the equation to the curve be $y^3 = ax^2 + x^3$.

Here $3y^2\dot{y} = 2ax\dot{x} + 3x^2\dot{x}$; $\therefore \dfrac{y\dot{x}}{\dot{y}} = \dfrac{3y^3}{2ax + 3x^2} =$

$\dfrac{3ax^2 + 3x^3}{2ax + 3x^2} = MT$. Hence $AT = \dfrac{3ax^2 + 3x^3}{2ax + 3x^2} - x = \dfrac{ax^2}{2ax + 3x^2}$.

Let x become infinite, $CL = \dfrac{3x^3}{3x^2} = x$; $Lx = \sqrt[3]{ax^2 + x^3} =$

x; $AC = \dfrac{ax^2}{3x^2} = \dfrac{a}{3}$. Therefore, since $CL : Lx :: CA :$

AR; $x : x :: \dfrac{a}{3} : AR = \dfrac{a}{3}$.

Hence, draw AR perpendicular to AC, and $= AC = \dfrac{a}{3}$,

and join CR. CR produced indefinitely will be the asymptote.

CHAP. V.

ON THE METHOD OF FINDING FLUENTS.

(37.) By the *direct* method of fluxions, we are taught how to find a fluxion from a fluent. The object of the *inverse* method, which is deduced from the former, is to find the fluent from the fluxion. In the former case, general rules are proposed, which are easy in the application; but it is frequently difficult to determine the fluent of a given fluxion, and in some cases even impossible; for it is obvious, that certain fluxions may be of such a nature as could not result from taking the fluxion of any fluent whatever. Rules can only be proposed for finding the fluents of those fluxions, whose forms prove them to have been deduced from some fluents.

(38.) To find the fluent of any power of a simple quantity which is multiplied by the fluxion of that quantity.

The fluxion of x^4 is $4x^3\dot{x}$; ∴ the fluent of $4x^3\dot{x}$ is x^4.

The fluxion of x^6 is $6x^5\dot{x}$; ∴ the fluent of $6x^5\dot{x}$ is x^6. And, in general, the fluxion of x^n is $nx^{n-1}\dot{x}$; ∴ the fluent of $nx^{n-1}\dot{x}$ is x^n.

Hence, to find the fluent, we have the following

RULE. Divide by the fluxion of the root, add 1 to the index, and divide by the index thus increased.

EXAMPLES.

Ex. 1. The fluent of $10x^9\dot{x} = \dfrac{10x^{9+1}\dot{x}}{10\dot{x}} = x^{10}$.

Ex. 2. The fluent of $40x^4\dot{x} = 8x^5$.

Ex. 3.   The fluent of $3x^7\dot{x} = \dfrac{3x^8}{8}$.

Ex. 4.   The fluent of $5x^{\frac{5}{3}}\dot{x} = \dfrac{5x^{\frac{8}{3}}}{\frac{8}{3}} = \dfrac{15x^{\frac{8}{3}}}{8}$.

Ex. 5.   The fluent of $\dfrac{4\dot{x}}{3x^{\frac{1}{2}}}$ or $\dfrac{4\dot{x} \times x^{-\frac{1}{2}}}{3} = \dfrac{4x^{\frac{1}{2}}}{\frac{1}{2} \times 3} = \dfrac{8x^{\frac{1}{2}}}{3}$.

Ex. 6.   The fluent of $\dfrac{3x^2\dot{x}}{2x^{\frac{1}{4}}}$ or of $\dfrac{3x^{\frac{3}{4}}\dot{x}}{2} = \dfrac{3x^{\frac{7}{4}}}{2 \times \frac{7}{4}} = x^{\frac{7}{4}}$.

(39.) If the quantity be compound, its fluent in certain cases may be found by the same Rule.

The fluxion of $(a^2 + x^2)^3 = 6x\dot{x}(a^2 + x^2)^2$; $\therefore$ the fluent of $6x\dot{x}(a^2 + x^2)^2$ is $(a^2 + x^2)^3$; and it is obtained by the Rule preceding, in the following manner; $a^2 + x^2$ is called the root; its fluxion is $2x\dot{x}$; divide by the fluxion of the root, and the result is $3(a^2 + x^2)^2$; add 1 to the index, and divide by it; the answer is $(a^2 + x^2)^3$; $\therefore$ in this case the Rule gives the correct fluent. And, in general, since the fluxion of $(a^n + x^n)^p = pnx^{n-1}\dot{x}(a^n + x^n)^{p-1}$, and that the Rule already laid down will from this fluxion deduce the right fluent $(a^n + x^n)^p$, it follows that the Rule holds in all fluxions of a similar nature; that is, wherever the index of the variable quantity without the vinculum is less by 1 than the index under the vinculum.

EXAMPLES.

Ex. 1.   The fluent of $(a^2 + x^2)^3 \ 6x\dot{x} = \dfrac{(a^2 + x^2)^4 \ 6x\dot{x}}{2x\dot{x} \times 4} = $
$(a^2 + x^2)^4 \times \dfrac{6}{8} = \dfrac{3}{4}(a^2 + x^2)^4$.

Ex. 2.   The fluent of $(a - {}^5x^5)^{\frac{3}{4}} \ x^4\dot{x} = \dfrac{(a^5 - x^5)^{\frac{7}{4}} \ x^4\dot{x}}{-5x^4\dot{x} \times \frac{7}{4}} = $
$-\dfrac{4}{35}(a^5 - x^5)^{\frac{7}{4}}$.

Ex. 3. The fluent of $\dfrac{4x^3\dot{x}}{(a^4+x^4)^{\frac{1}{2}}}$, or $(a^4+x^4)^{-\frac{1}{2}} 4x^3\dot{x} =$

$\dfrac{(a^4+x^4)^{\frac{1}{2}} 4x^3\dot{x}}{4x^3\dot{x} \times \frac{1}{2}} = 2 (a^4+x^4)^{\frac{1}{2}}.$

Ex. 4. The fluent of $(a^6 + 4x^6)^{\frac{2}{7}} x^5\dot{x} = \dfrac{(a^6 + 4x^6)^{3} x^5\dot{x}}{24x^5\dot{x} \times \frac{5}{7}}$

$= \dfrac{(a^6+4x^6)^{\frac{5}{7}}}{60}.$

Ex. 5. To find the fluent of $(a+x)^4 \dot{x}$.

Since $x^0 = 1$, this is the same as $(a+x)^4 x^0\dot{x}$; and the index of x without the vinculum is one less than the index of x under the vinculum; therefore the Rule applies, and the fluent $= \dfrac{(a+x)^5 \dot{x}}{\dot{x} \times 5} = \dfrac{(a+x)^5}{5}.$ In the same manner,

Ex. 6. The fluent of $(9y+4a)^{\frac{1}{2}} \dot{y} = \dfrac{2}{27} (9y+4a)^{\frac{3}{2}}.$

(40.) If all the quantities under the vinculum be variable, and the quantity without be in any given ratio to the fluxion of the root, the fluent may be found as before.

Ex. 1. Let $(x^2 + y^2)^3 (6x\dot{x} + 6y\dot{y})$ be the fluxion, whose fluent is required; the root is $x^2 + y^2$, and its fluxion $= 2x\dot{x} + 2y\dot{y}$;

$\therefore$ the fluent $= \dfrac{(x^2 + y^2)^4 (6x\dot{x} + 6y\dot{y})}{(2x\dot{x} + 2y\dot{y}) 4} = \dfrac{3}{4} (x^2+y^2)^4.$

Ex. 2. The fluent of $(x^4 + y^6 + z^8)^{\frac{1}{2}} (8x^3\dot{x} + 12y^5\dot{y} + 16z^7\dot{z}) = \dfrac{(x^4 + y^6 + z^8)^{\frac{3}{2}} (8x^3\dot{x} + 12y^5\dot{y} + 16z^7\dot{z})}{(4x^3\dot{x} + 6y^5\dot{y} + 8z^7\dot{z}) \times \frac{3}{2}} =$

$\dfrac{(x^4+y^6+z^8)^{\frac{3}{2}} 4}{3}.$

Ex. 3. The fluent of $(a^2x^2+x^4)^{\frac{1}{2}}\ (2a^2x\dot{x}+4x^3\dot{x}) = \frac{2}{3}$

$(a^2x^2+x^4)^{\frac{3}{2}}$.

TO FIND FLUENTS BY LOGARITHMS.

(41.) Let y be any number, and x its logarithm; then if x increase uniformly, or in arithmetic progression, by the nature of logarithms y increases in geometric; but if quantities increase in geometric progression, their differences or increments are proportional to the quantities themselves; that is, ultimately, $y \propto \dot{y}$, and $\dfrac{\dot{y}}{y}$ is constant;

but $\dot{x}$ is constant, $\therefore \dfrac{\dot{y}}{y} \propto \dot{x}$; and if m be assumed of a

proper magnitude, $\dot{x} = \dfrac{m\dot{y}}{y}$; or the fluxion of any logarithm is equal to some constant quantity multiplied into the fluxion of the number, and this product divided by the number itself.

This quantity m is called the modulus of the system.

If $m = 1$, $\dot{x} = \dfrac{\dot{y}}{y}$, an equation which may be deduced from the hyperbola; hence these logarithms are called hyperbolic.

Cor. If $x =$ the hyperbolic logarithm of y, $\dot{x} = \dfrac{\dot{y}}{y}$;

and conversely, if $\dot{x} = \dfrac{\dot{y}}{y}$, $x =$ hyp. log. of y. Now y may represent any compound number, and $\dot{y}$ the fluxion of that number; hence, if any fluxional expression consist of the fluxion of a quantity divided by that quantity itself, the fluent will be the hyperbolic logarithm of the quantity.

EXAMPLES.

(42.) **Ex. 1.** The fluent of $\dfrac{\dot{x}}{1+x}$ = hyp. log. $(1+x.)$

Ex. 2. The fluent of $\dfrac{2x\dot{x}}{a^2+x^2}$ = hyp. log. $(a^2+x^2.)$

Ex. 3. The fluent of $\dfrac{bx^2\dot{x}}{a^3+x^3}$ = $\dfrac{b}{3}$ × fluent of $\dfrac{3x^2\dot{x}}{a^3+x^3}$

= $\dfrac{b}{3}$ × hyp. log. $(a^3+x^3.)$

Ex. 4. The fluent of $\dfrac{x^{n-1}\dot{x}}{a^n+x^n}$ = $\dfrac{1}{n}$ × fluent of $\dfrac{nx^{n-1}\dot{x}}{a^n+x^n}$

= $\dfrac{1}{n}$ × hyp. log. $(a^n+x^n.)$

Ex. 5. The fluent of $\dfrac{y\dot{y}}{1+\dfrac{4y^2}{a^2}}$ = fluent of $\dfrac{a^2}{8}$ × $\dfrac{2y\dot{y}}{\dfrac{a^2}{4}+y^2}$

= $\dfrac{a^2}{8}$ × hyp. log. $\left(\dfrac{a^2}{4}+y^2.\right)$

(43). In these and similar cases, the application of the Rule is obvious; but as these instances do not often occur, it is generally necessary, in fluxional expressions which admit fluents by logarithms, to use one of the following forms.

FORM I. The fluent of $\dfrac{\dot{x}}{\sqrt{x^2\pm a^2}}$ = hyp. log. $\overline{x+\sqrt{x^2\pm a^2}}$.

Let $\sqrt{x^2\pm a^2}=v$, then $(x^2\pm a^2)=v^2$, and $x\dot{x}=v\dot{v}$;

$\therefore\ \dot{x}\ :\ \dot{v}\quad ::\ v\ :\ x,$

and $\dot{x}\ :\ \dot{x}+\dot{v}\ ::\ v\ :\ v+x;\ \therefore\ \dfrac{\dot{x}}{v}=\dfrac{\dot{x}+\dot{v}}{x+v}$, or $\dfrac{\dot{x}}{\sqrt{x^2\pm a^2}}$

= $\dfrac{\dot{x}+\dot{v}}{x+v}$; $\therefore$ the fluent of $\dfrac{\dot{x}}{\sqrt{x^2\pm a^2}}$ = hyp. log. $(x+v)$

= hyp. log. $(x+\sqrt{x^2\pm a^2}.)$

FORM II. The fluent of $\dfrac{\dot{x}}{\sqrt{x^2 \pm 2ax}}$ is the hyp. log. of

$(x \pm a + \sqrt{x^2 \pm 2ax})$.

Let $\sqrt{x^2 \pm 2ax} = v$; then $x^2 \pm 2ax + a^2 = v^2 + a^2$,

and $x \pm a = \sqrt{v^2 + a^2}$; $\therefore \dot{x} = \dfrac{v\dot{v}}{\sqrt{v^2 + a^2}}$, and

$\dfrac{\dot{x}}{\sqrt{x^2 \pm 2ax}} = \dfrac{\dot{v}}{\sqrt{v^2 + a^2}}$; $\therefore$ the fluent of $\dfrac{\dot{x}}{\sqrt{x^2 \pm 2ax}}$

$=$ fluent of $\dfrac{\dot{v}}{\sqrt{v^2 + a^2}} = $ hyp. log. $(v + \sqrt{v^2 + a^2}) = $

hyp. log. of $(x \pm a + \sqrt{x^2 \pm 2ax})$.

FORM III. The fluent of $\dfrac{2a\dot{x}}{a^2 - x^2}$ is the hyp. log. of

$\dfrac{a + x}{a - x}$.   Assume $\dfrac{A}{a + x} + \dfrac{B}{a - x} = \dfrac{2a}{a^2 - x^2}$; then

$\dfrac{Aa - Ax + Ba + Bx}{a^2 - x^2} = \dfrac{2a}{a^2 - x^2}$, or $\left\{ \begin{array}{c} Aa + Bx \\ + Ba - Ax \\ - 2a \end{array} \right\}$

$= 0$; hence, equating the homologous terms, $Bx - Ax$
$= 0$; $\therefore B = A$;

also, $Aa + Ba - 2a = 0$, or $2Aa = 2a$; $\therefore A = 1$, and

$B = 1$; $\therefore \dfrac{2a\dot{x}}{a^2 - x^2} = \dfrac{\dot{x}}{a + x} + \dfrac{\dot{x}}{a - x} = \dfrac{\dot{x}}{a + x} - 1 \times$

$\dfrac{-\dot{x}}{a - x}$, and the fluent $=$ hyp. log. $(a + x) - $ hyp. log.

$(a - x) = $ hyp. log. $\dfrac{a + x}{a - x}$.

In the same manner the fluent of $\dfrac{2a\dot{x}}{x^2 - a^2} = $ hyp. log.

$\dfrac{x - a}{x + a}$.

FORM IV. The fluent of $\dfrac{2a\dot{x}}{x\sqrt{a^2+x^2}}$ is the hyp. log. of

$\dfrac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}$.

Let $\sqrt{a^2+x^2}=v$; then $a^2+x^2=v^2$; and $x\dot{x}=v\dot{v}$;

$\therefore \dfrac{2a\dot{x}}{xv}=\dfrac{2a\dot{v}}{x^2}$, or $\dfrac{2a\dot{x}}{x\sqrt{a^2+x^2}}=\dfrac{2a\dot{v}}{v^2-a^2}$; and the

fluent of $\dfrac{2a\dot{x}}{x\sqrt{a^2+x^2}}=$ hyp. log. $\dfrac{v-a}{v+a}=$ hyp. log. of

$\dfrac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}$.

In the same manner, the fluent of $\dfrac{2a\dot{x}}{x\sqrt{a^2-x^2}}=$ hyp.

log. $\dfrac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}$.

These are the principal forms of fluxions, whose fluents may be found by a table of hyperbolic logarithms. This table may be supplied by a table of the common form; for the hyperbolic logarithm of any number : the common log. :: 1 : m the modulus of the common system. This subject is more fully explained under the Article LOGARITHMS.

(44.) To find fluents by means of circular arcs. Let AB be a circular arc, whose center is C; BD the right sine, AT the tangent, CT the secant; let Bn, nm represent the fluxion of AD and BD; then Bm is the fluxion of the arc. For Bn and nm being described uniformly, Bm is described uniformly, and is in the direction of a tangent at B; also, since
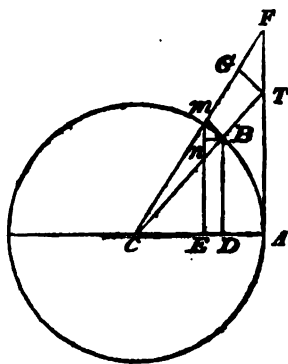
the arc is described by a velocity either continually acce-
lerated or continually retarded, *Bm* is a limit between
the increments. Conceive *Bn* and *nm* to be very small.
Join *Cm*, and let it meet *AT* produced in *F*; draw *TG*
perpendicular to *CF*. Assume $CA = a$, $AD = x$,
$AB = z$, $DB = y$, $AT = t$, $CT = s$, $Bn = \dot{x}$, $nm = \dot{y}$,
$Bm = \dot{z}$; and since ultimately the fluxion = the incre-
ment, $TF = \dot{t}$, and $GF = \dot{s}$.

CASE I. To find the fluxion of a circular arc in terms
of the right sine.

By the similar triangles *Bmn, CBD,*
$Bm : mn :: CB : CD$

or $\dot{z} : \dot{y} :: a : \sqrt{a^2 - y^2}$; $\therefore \dot{z} = \dfrac{a\dot{y}}{\sqrt{a^2 - y^2}}$.

CASE II. To find the fluxion of a circular arc in terms
of the versed sine.

By the same triangles,
$Bm : Bn :: CB : BD$

or $\dot{z} : \dot{x} :: a : \sqrt{2ax - x^2}$;

$\therefore \dot{z} = \dfrac{a\dot{x}}{\sqrt{2ax - x^2}}$.

CASE III. To find the fluxion of a circular arc in terms
of the tangent.

By the similar triangles *CBm, CTG,*
$Bm : TG :: CB : CT$. And by sim. tri^s. *FTG, ACT,*
$TG : TF :: AC : CT$;
$\therefore Bm : TF :: AC \times CB : CT^2$; or
$\dot{z} : \dot{t} :: a^2 : a^2 + t^2$;

$\therefore \dot{z} = \dfrac{a^2\dot{t}}{a^2 + t^2}$.

CASE IV. To find the fluxion of a circular arc in terms
of the secant.

By the same triangles,

$Bm : TG :: CB : CT.$ And

$TG : GF :: CA : AT$;

$\therefore Bm : GF :: AC \times BC : AT \times CT$; or

$$\dot{z} : \dot{s} :: a^2 : \sqrt{s^2 - a^2} \times s;$$

$$\therefore \dot{z} = \frac{a^2 \dot{s}}{s \times \sqrt{s^2 - a^2}}.$$

These are the four principal forms. But it may be useful to add another.

CASE V. To find the fluxion of a circular arc in terms of the cosine.

In this case let $CD = x$, $Bn = -\dot{x}$; for as the arc increases CD diminishes; and $BD = \sqrt{a^2 - x^2}$.

Hence, by similar triangles,

$Bm : Bn :: CB : BD$;

or $\dot{z} : -\dot{x} :: a : \sqrt{a^2 - x^2}$;

$$\therefore \dot{z} = \frac{-a\dot{x}}{\sqrt{a^2 - x^2}}.$$

COR. 1. The fluent of $\dfrac{a\dot{y}}{\sqrt{a^2 - y^2}} = \dot{z} = $ a circular arc, whose radius is a, and sine y.

COR. 2. The fluent of $\dfrac{a\dot{x}}{\sqrt{2ax - x^2}}$ is a circular arc, of radius a, and versed sine x.

COR. 3. The fluent of $\dfrac{a^2\dot{t}}{a^2 + t^2}$ is a circular arc, whose radius is a, and tangent t.

COR. 4. The fluent of $\dfrac{a^2\dot{s}}{s \times \sqrt{s^2 - a^2}}$ is a circular arc, whose radius is a, and secant s.

COR. 5. The fluent of $\dfrac{-a\dot{x}}{\sqrt{a^2 - x^2}}$ is a circular arc whose radius is a, and cosine x.

(45.) Hence, since the circumferences of circles and corresponding arcs are as their radii, if we know the length of an arc to any one radius, we may find it for any other. Let $A = $ the length of an arc to radius 1 ; required to find the length of an arc subtending the same angle to radius a; the proportion is $1 : a :: A :$ the arc required $= a \times A$. Thus, if the length of an arc of $30°$ to radius $1 = 0{,}5235987$, the length of the corresponding arc to radius $a = a \times 0{,}5235987$.

------▸------

ON THE CORRECTION OF FLUENTS.

(46.) The fluxion of x is $\dot{x}$; and the fluxion of $x \pm a$ is $\dot{x}$, whatever be the value of the constant part a, and whatever its sign. Under different circumstances, therefore, the fluent of $\dot{x}$ may be either x, or $x \pm a$. So that though a fluent can only have one fluxion, yet a fluxion, in different cases, may have different fluents ; and this must be determined from the nature of the problem. First, take the fluent according to the rules, and observe whether this fluent becomes equal to nothing, or to some determined value, when the nature of the problem requires that it should ; if it do, no constant quantity is to be annexed to it ; if not, some constant quantity must be added or subtracted to make it $= 0$, or to give it the value required. This is called the correction of a fluent. Instances will be given in the following sections.

CHAP. VI.

ON THE AREAS OF CURVES.

(47.) THE fluxion of the area of an algebraic curve is equal to the rectangle contained by the ordinate and the fluxion of the abscissa.

Let AMP be any curvilinear area, generated by the
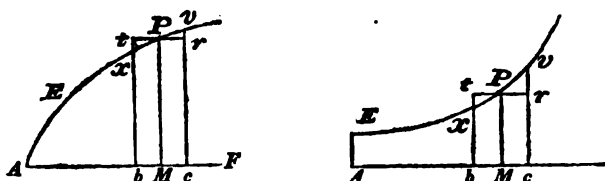
uniform motion of the ordinate PM. Take Mb and Mc on each side of MP, and equal to each other; and let tPr, parallel to AF, meet bx and cv, which are drawn parallel to MP, in t and r. Then, since the abscissa AM is supposed to flow uniformly, if the ordinate MP be conceived to increase, either of the equal parallelograms $PtbM$, $PrcM$, is greater than the preceding increment $PxbM$, and less than the succeeding $PvcM$. Also, since $PM = rc$, the rectangle $PMcr$ increases or decreases according to the increase or decrease of Mc; therefore $PMcr$ is the fluxion of the area AMP. Hence, if $AM = x$, $PM = y$, the fluxion of the area $APM = y\dot{x}$.

EXAMPLES.

(48.) Ex. 1. To find the area of a triangle ABC. Draw AD perpendicular, and EF parallel to BC. Let $AD = a$, $BC = b$, $AG = x$, $EF = y$; then, by similar triangles, $a : b :: x : y$; $\therefore y = \dfrac{bx}{a}$;
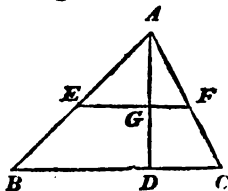
$\therefore y\dot{x}$, the fluxion of the area $= \dfrac{bx\dot{x}}{a}$, and the fluent $=$

$\dfrac{bx^2}{2a} = \dfrac{y \times x}{2}$; that is, the area $ABC = \dfrac{BC \times DA}{2}$.

Ex. 2. To find the area of the common parabola *APM.*

The equation is $ax = y^2$; $\therefore a\dot{x} = 2y\dot{y}$;

$\therefore y\dot{x} = \dfrac{2y^2\dot{y}}{a}$; and the fluent $= \dfrac{2y^3}{3a} =$

$\dfrac{2y^2 \times y}{3a} = \dfrac{2ax \times y}{3a} = \dfrac{2}{3}yx$, or the area

$AMP = \dfrac{2}{3} AM \times MP, = \dfrac{2}{3}$ of the circumscribing rectangle *AMPB.*
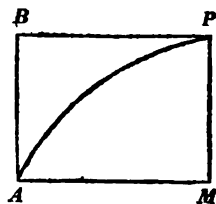
Ex. 3. In general, to find the area of any parabola;
The general equation is $a^{n-1}x = y^n$;

$\therefore a^{n-1}\dot{x} = ny^{n-1}\dot{y}$; $\therefore y\dot{x} = \dfrac{ny^n\dot{y}}{a^{n-1}}$; and the fluent $=$

$\dfrac{ny^{n+1}}{(n+1).a^{n-1}} = \dfrac{ny^n \times y}{(n+1).a^{n-1}} = \dfrac{na^{n-1}x \times y}{(n+1).a^{n-1}} = \dfrac{n \times xy}{n+1} =$

$\dfrac{n}{n+1} \times$ the circumscribing rectangle.

Cor. If $n = 2$, the area $= \dfrac{2}{3}$ of this rectangle, as before.

Ex. 4. To compare the areas of two parabolas described on the same axis, whose *latera recta* are L and l.

Let AP and Ap be the parabolas; AM the common axis. Then the fluxion of AMP : the fluxion of AMp :: $MP \times \dot{x}$: $Mp \times \dot{x}$
:: MP : Mp; but
$L \times AM = MP^2$; $\therefore \sqrt{L \times AM} = MP$;
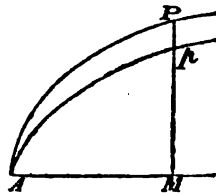
and $\sqrt{l \times AM} = Mp$; $\therefore$ the fluxion of AMP : the fluxion of $AMp :: \sqrt{L \times AM} : \sqrt{l \times AM} :: \sqrt{L} : \sqrt{l}$. But quantities, whose fluxions are in constant ratio, are themselves in the same constant ratio; $\therefore$ the area AMP : the area $AMp :: \sqrt{L} : \sqrt{l}$.

Ex. 5. To compare the area of an ellipse, whose major axis is $2a$, and minor $2b$, with the area of a circle, whose diameter is the major axis of the ellipse.

Let APD be the circle, ApD the ellipse, AM an abscissa common to both, PM and pM the corresponding ordinates, CB half the minor axis of the ellipse, and CE the radius of the circle $= CA$.
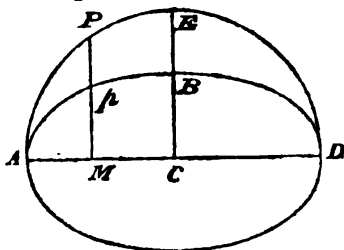
Then the fluxion of AMP : the fluxion of $AMp :: MP$: $Mp :: CE : CB$ (by the nature of the ellipse) $:: CA$: $CB :: 2CA : 2CB$.

$::$ the major axis : the minor, which is a given ratio; $\therefore$ the area AMP : the area $AMp ::$ the major axis : the minor; and $ACE : ACB$, or the whole area of the circle : the whole area of the ellipse $::$ the major : the minor.

(49.) The same proposition may be easily proved by means of the following Lemma. (See the FIGURE above.)

If APD be a circle, AD the diameter $= 2a$, and PM an ordinate, the area $AMP =$ the fluent of $\dot{x} \times \sqrt{2ax - x^2}$. For PM or $y = \sqrt{2ax - x^2}$; $\therefore y\dot{x}$ or the fluxion of the area $AMP = \dot{x} \times \sqrt{2ax - x^2}$; $\therefore$ the area AMP itself $=$ the fluent of $\dot{x} \times \sqrt{2ax - x^2}$.

Hence, to find the area of an ellipse, whose major axis is $2a$ and minor $2b$, in terms of the area of a circle whose diameter $= 2a$.

In the ellipse, $y = \dfrac{b}{a} \times \sqrt{2ax - x^2}$; $\therefore y\dot{x}$ or the fluxion of the area AMp (See Fig. to Ex. 5.) $= \dfrac{b}{a} \times \dot{x} \times \sqrt{2ax - x^2}$;

$\therefore$ the fluent or area $AMp = \dfrac{b}{a} \times$ the fluent of $\dot{x} \times$

$\sqrt{2ax - x^2} = \dfrac{b}{a} \times$ the circular area AMP, and the whole

area of the ellipse $ABD = \dfrac{b}{a} \times$ the area of the circle

AED; therefore,
the area of the ellipse : area of the circle on major axis
:: $b : a$, or :: the minor axis : the major.

Ex. 6. Let DEF be an hyperbola, of which the asymptotes are CM and CN; to find the area $EGHF$ comprehended between two ordinates EG, FH.

Let $CG = a$, $GE = b$, $GH = x$, $HF = y$; then, by the nature of the hyperbola, $CG \times GE = CH \times HF$, or $a \times b = (a + x) \, y$; $\therefore y = \dfrac{ab}{a+x}$, and $y\dot{x} = \dfrac{ab \times \dot{x}}{a+x}$; $\therefore$ the fluent $= ab \times$ hyp. log. $(a + x) +$ cor.
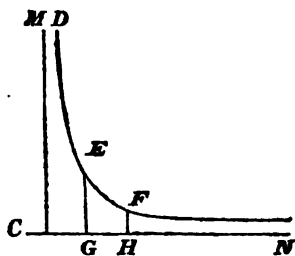
Let the area $= 0$, $x = 0$; $\therefore$ cor. $= -ab \times$ hyp. log. a;

$\therefore$ the correct area $EGHF = ab \times$ hyp. log. $\dfrac{a+x}{a}$.

Cor. If CG and GE each $= 1$, $y\dot{x} = \dfrac{\dot{x}}{1 + x}$; and the fluent $=$ hyp. log. $(1 + x) +$ cor. $=$ hyp. log. $(1 + x) -$ hyp. log. 1; but hyp. log. of $1 = 0$; $\therefore EGFH =$ hyp. log. $(1 + x)$; or the area $EGHF$ is the hyperbolic logarithm of the abscissa; the modulus here $= 1$.

Ex. 7. To find the area of a cycloid,

Let CAL be a cy-
cloid, AD the axis,
ABD the generating
circle, AF a tangent
at the vertex, CF pa-
rallel to AD. Take
any point P in the arc, and draw PM perpendicular
to AM. Then the fluxion of the external area $AMP =$
$PM \times$ the fluxion of AM. Let $AE = x$, $AB = 2a$;
then $BE = \sqrt{2ax - x^2}$, and the fluxion of $BE =$
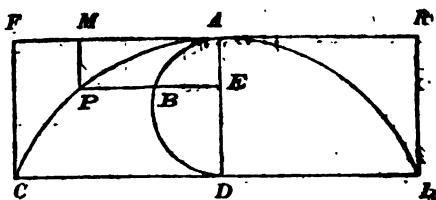
$$\frac{ax - xx}{\sqrt{2ax - x^2}}.$$

Also $PB =$ the arc BA; $\therefore$ the fluxion of $PB =$

$$\frac{ax}{\sqrt{2ax - x^2}};$$ $\therefore$ the fluxion of $PB + BE$, or of $AM =$

$$\frac{2ax - xx}{\sqrt{2ax - x^2}};$$ and the fluxion of the area, APM or $PM \times$

the fluxion of $AM = \dfrac{2axx - x^2x}{\sqrt{2ax - x^2}} = x \times \dfrac{2ax - x^2}{\sqrt{2ax - x^2}} =$

$x \times \sqrt{2ax - x^2}$; but $x \times \sqrt{2ax - x^2}$ (Art. 49.) $=$ the
fluxion of AEB; $\therefore$ the fluxion of $APM =$ the fluxion
of AEB; and these two areas begin together; $\therefore APM$
$= ABE$; and the whole external area $ACF =$ the semi-
circle ABD; for the same reason, $ALR =$ the other
semicircle; $\therefore$ the whole area, without the cycloid, $=$
the generating circle; but the parallelogram $CERL =$
$CL \times DA =$ the circumference $\times$ the diameter $= 4$
generating circles; $\therefore$ the area of the cycloid $= 3$ gene-
rating circles.

The same conclusion may be obtained in the following
manner. Draw pm, pe indefinitely near and parallel
to PM, PE, and produce EP till it cuts pm in y. Join

AB, and draw *Bb* parallel to *Ee*. The increments *MPym*, *BEeb* represent the fluxions of the area *APM* and of the circular area *ABE*. Now, by similar triangles,

$$AE \;:\; EB \;::\; py \;:\; yP,$$
$$\text{or} \quad MP \;:\; EB \;::\; Ee \;:\; yP;$$

$\therefore MP \times yP = EB \times Ee$, or the fluxion of *APM*=the fluxion of *AEB*; and these two areas begin together; $\therefore$ &c., as before.

Ex. 8. To find the area of a curve, whose equation is

$$y = \frac{ax}{\sqrt{ax - x^2}}.$$

Here $y\dot{x} = \dfrac{ax\dot{x}}{\sqrt{ax - x^2}}$; whose fluent is (Fluent 23.) $a \times$ a circular arc of radius $\frac{1}{2}a$, and versed sine $x - a \times \sqrt{ax - x^2}$, which vanishes when $x = 0$; and if $x = a$, it becomes $a \times$ the semi-circumference, whose radius $= \frac{1}{2}a =$ the area of two circles of radius $= \frac{1}{2}a$.

Ex. 9. The area of a curve equals n times its circumscribing rectangle; required the equation.

Here the fluent of $y\dot{x} = n \times yx$; $\therefore y\dot{x} = ny\dot{x} + nx\dot{y}$; $\therefore (1-n) y\dot{x} = nx\dot{y}$; $\therefore (1-n) \dfrac{\dot{x}}{x} = n \times \dfrac{\dot{y}}{y}$; and $(1-n)$ hyp. log. $x = n \times$ hyp. log. y; or $x^{1-n} \propto y^n$.

In the common parabola, $n = \dfrac{2}{3}$; $\therefore x^{\frac{1}{3}} \propto y^{\frac{2}{3}}$, or $x \propto y^2$, and $ax = y^2$ the equation.

Ex. 10. Required the nature of a curve whose area$=$ hyp. log. $\dfrac{a+x}{a-x}$.
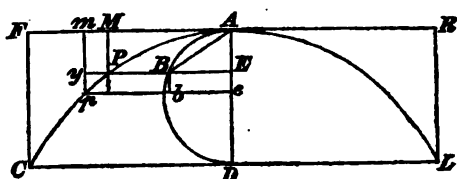
Here the fluent of $y\dot{x} = $ hyp. log. $\dfrac{a+x}{a-x} = $ hyp. log.

$a+x-$ hyp. log. $a-x$; $\therefore y\dot{x} = \dfrac{\dot{x}}{a+x} + \dfrac{\dot{x}}{a-x} = \dfrac{2a\dot{x}}{a^2-x^2}$;

and $y = \dfrac{2a}{a^2-x^2}$ the equation.

Ex. 11. Required the nature of the curve, the square of whose ordinate BC is a mean proportional between some constant quantity a^2, and the curvilinear area ABC.
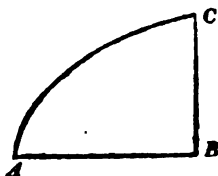
By the problem,
the fluent of $y\dot{x} : y^4 :: y^2 : a^2$;

$\therefore a^2 \times$ the fluent of $y\dot{x} = y^4 \propto a^2y\dot{x} = 4y^3\dot{y}$; $\therefore a^2\dot{x} = 4y^2\dot{y}$,

and $a^2x = \dfrac{4y^3}{3}$; $\therefore x \propto y^3$.

D

Chap. VII.

TO FIND THE CONTENTS OF SOLIDS.

(50.)   **L**et *PAF* represent a solid, which is conceived
to be generated by a circle,
beginning its motion at *A*,
and perpetually changing its
magnitude, while its centre
moves uniformly on the line
AM. Let *PRF* denote the
position of this circle, cor-
responding to the ordinate
PM of the curve *AP.*
Through *P* and *F* draw two lines, *tPr, eFp,* parallel to
AC; take *Mb = Mc,* and through *b* and *c* draw two
lines parallel to *PF,* meeting *tP r, eFp,* in *t, e, r* and *p.*
Conceive circles to be described on the diameters *te, xd,*
vg and *rp.* Then the cylinder *PrpF = PteF*; and
PrpF is less than *PvgF*; and its equal *PteF* is greater
than *PxdF*; that is, the cylinder *PrpF* is greater than
the preceding, and less than the succeeding increment of
the solid. Also, since *PF* is constant in the cylinder
PFpr, the increase or decrease of this cylinder will vary
as the increase or decrease of *MC*; but *AM* flows
uniformly, ∴ the cylinder *PFp r* is the fluxion of the
solid *PAF.*
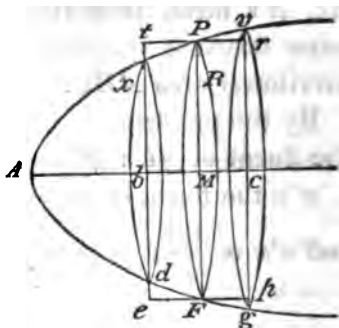
Let $AM = x$; $PM = y$; $MC = \dot{x}$; then, if $p =$
3.14159, &c. the area of a circle, whose radius is 1, the con-
tent of this cylinder, or the fluxion of the solid $= p y^2 \dot{x}$.

Cor.   Whatever be the form of the generating plane,

as a triangle, a square, a parallelogram, &c. the fluxion of the solid will be equal to the product of this area and the fluxion of the abscissa.

EXAMPLES.

(51.) Ex. 1. Let ABE be the solid generated by the revolution of the common parabola about its axis.

Here $y^2 = ax$; $\therefore py^2\dot{x} = pax\dot{x}$,

and the fluent or content $= \dfrac{pax^2}{2} +$

correct.; let $x = 0$; then the content $= 0$, $\therefore C = 0$; or the content $=$

$\dfrac{pax^2}{2} = \dfrac{pax \times x}{2} = \dfrac{py^2 \times x}{2}$; that is,

the whole content of $ABE = p \times \dfrac{BC^2 \times AC}{2}$.
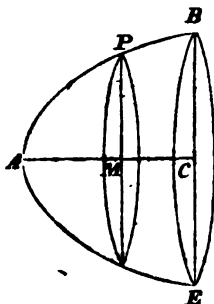
Cor. Since $p \times BC^2 \times AC =$ the content of the circumscribing cylinder, the paraboloid $=$ half the circumscribing cylinder.

Ex. 2. Let ABE be the solid, generated by the revolution of any parabola about its axis.

In this case, $a^{n-1}x = y^n$; $\therefore a^{n-1}\dot{x} = ny^{n-1}\dot{y}$;

$\therefore py^2\dot{x} = \dfrac{npy^{n+1}\dot{y}}{a^{n-1}}$, and the fluent or content $=$

$\dfrac{npy^{n+2}}{(n+2)a^{n-1}} +$ correction. But if $y = 0$, the content

$= 0$, $\therefore$ cor. $= 0$; $\therefore$ the content $= \dfrac{npy^{n+2}}{(n+2)a^{n-1}} =$

$\dfrac{npy^n \times y^2}{(n+2)a^{n-1}} = \dfrac{npa^{n-1}xy^2}{(n+2)a^{n-1}} = \dfrac{n}{n+2} \times py^2x$.

Cor. The content of ABE in this case $= \dfrac{n}{n+2} \times$ the content of the circumscribing cylinder.

If $n = 2$, the paraboloid $= \frac{1}{2}$ the cylinder as before.

Ex. 3. To find the content of a sphere.

Let the radius $= a$, the versed sine $= x$, and the right sine y; then the equation is $y^2 = 2ax - x^2$, $\therefore p\dot{y}^2\dot{x} =$

$p\,(2ax\dot{x} - x^2\dot{x})$; and the fluent $= p\left(ax^2 - \dfrac{x^3}{3}\right)$ + cor.

Let $x = 0$; then the sphere $= 0$; $\therefore$ cor. $= 0$; and the

fluent $= p\left(ax^2 - \dfrac{x^3}{3}\right)$; let $x = 2a$; then the whole

content $= p\left(4a^3 - \dfrac{8a^3}{3}\right) = \dfrac{4pa^3}{3}$.

Cor. The content of a cylinder circumscribing the sphere $= pa^2 \times 2a = 2pa^3$; therefore,

the sphere : its circumscribing cylinder :: $\dfrac{4pa^3}{3}$: $2pa^3$,

$$:: \quad 2 \quad : \quad 3.$$

Ex. 4. To find the content of a spheroid generated by the revolution of a semi-ellipse about the axis major.

Here $y^2 = \dfrac{b^2}{a^2}(2ax - x^2)$;

$\therefore p\dot{y}^2\dot{x} = \dfrac{pb^2}{a^2}(2ax\dot{x} - x^2\dot{x})$;

$\therefore$ by the last case, the con-
tent of the section correspond-

ing to the abscissa $x = \dfrac{pb^2}{a^2}\left(ax^2 - \dfrac{x^3}{3}\right)$; the cor. $= 0$;

and the content of the whole spheroid $= \dfrac{pb^2}{a^2} \times \dfrac{4a^3}{3} =$

$\dfrac{4pb^2a}{3}$.
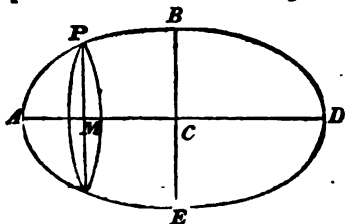
Cor. 1. If the ellipse revolve round the minor instead of the major axis, since the same property of the curve obtains in each case, the content of the spheroid thus

generated $= \frac{4p a^2 b}{3}$. Hence the solid generated round the major : the solid generated round the minor :: $\frac{4p b^2 a}{3}$:

$\frac{4p a^2 b}{3}$ :: $b : a$ :: $BC : CA.$

COR. 2. Since $\frac{4p b^2 a}{3} : \frac{4p a^2 b}{3} :: \frac{4p a^2 b}{3} : \frac{4p a^3}{3}$; $\therefore$ the solid generated round the minor is a mean proportional between the solid generated round the major and the sphere, whose diameter is equal to the major axis.

Ex. 5. To find the content of an hyperboloid.
Let BAE (See the FIGURE in the preceding page) represent a hyperboloid, whose major axis $= 2a$, minor $= 2b$, $AM = x$, $MP = y$, $AC = e$, then $y^2 = \frac{b^2}{a^2}$

$(2ax + x^2)$; $\therefore py^2 \dot{x} = \frac{p b^2}{a^2} (2ax\dot{x} + x^2\dot{x})$; and the

fluent $= \frac{p b^2}{a^2} \left(ax^2 + \frac{x^3}{3} \right)$, which needs no correction;

or the whole content $= \frac{p b^2}{a^2} \left(ae^2 + \frac{e^3}{3} \right).$

COR. The content of a cylinder, the radius of whose base is PM, and altitude $AM = p \times PM^2 \times AM = \frac{p b^2}{a^2}$

$(2ax^2 + x^3)$; $\therefore$ the content of the hyperboloid of the altitude x : the content of a cylinder of the same base and altitude :: $ax^2 + \frac{x^3}{3}$: $2ax^2 + x^3$. Let x be very small, and this becomes the ratio of ax^2 : $2ax^2$, or of $1 : 2$, which is the ratio in the paraboloid.

Ex. 6. To find the content of a cone $ABC.$

Let AD the axis $= a$, $DC = b$,
$AG = x$, $GF = y$. Then $y =$
$\dfrac{bx}{a}$; $\therefore py^2\dot{x} = \dfrac{pb^2x^2\dot{x}}{a^2}$, and the

fluent $= \dfrac{pb^2x^3}{3a^2}$, which needs no

correction. Let $x = a$; then the
content of the whole cone $=$

$\dfrac{pb^2a}{3} = \frac{1}{3}$ of a cylinder of the same base and altitude.
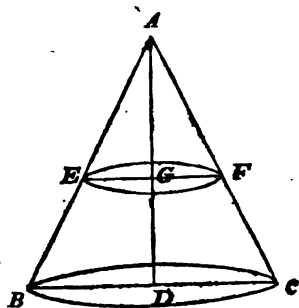
Ex. 7. To find the content of the solid called The
Groin, which is generated by a variable square, $mnpq$,
moving parallel to itself, the section GAK through the
middle of the opposite sides being semi-circular.

Let $AH = a$; AF
$= x$; $FI = y$; then
$y = \sqrt{2ax - x^2}$ by
the nature of the cir-
cle; $\therefore 2y = 2 \times$
$\sqrt{2ax - x^2}$; $\therefore$ the
area of the generating
plane $mnpq$, which answers to py^2 in the other cases,
$= 4 \times (2ax - x^2)$; hence $py^2\dot{x} = 8ax\dot{x} - 4x^2\dot{x}$; and the

fluent $= 4ax^2 - \dfrac{4x^3}{3} +$ correction, and the correction
$= 0$. Let $x = a$; then the whole solid $BAD = 4a^3$
$- \dfrac{4a^3}{3} = \dfrac{8a^3}{3}$.
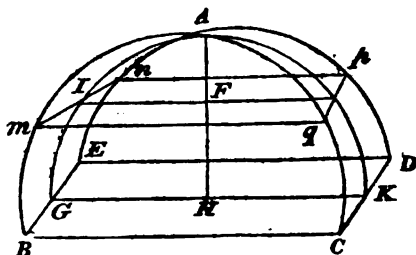
Ex. 8. To find the content of a pyramid, the section
of which parallel to the base is any
given figure.

Let EFG be a section parallel to
the base; draw AIK perpendicular
to the plane BDC, cutting EFG in
I; join KD, FI; then the area of
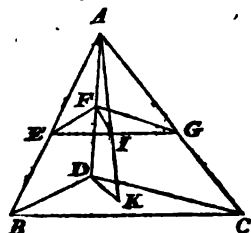
the base BDC : the area of EFG :: DC^2 : FG^2 :: AD^2 : AF^2 :: AK^2 : AI^2; hence if $A =$ the area of the base, $AK = a$, $AI = x$, the generating area, EFG $= \dfrac{Ax^2}{a^2}$; $\therefore$ $py^2\dot{x} = \dfrac{Ax^2\dot{x}}{a^2}$; and the fluent $= \dfrac{Ax^3}{3a^2}$, which needs no correction. Let $x = a$; then the content of the whole figure $= \dfrac{A \times a}{3} = \frac{1}{3}$ the content of a prismatic figure of the same base and altitude.

Chap. VIII.

ON THE LENGTHS OF CURVES.

(52.) Def. A curve is described by the motion of a point which is continually changing its direction.

Cor. 1. If MPN be a tangent to the curve AB at P, the direction in which the describing point moves, when at P, is in the tangent PN.
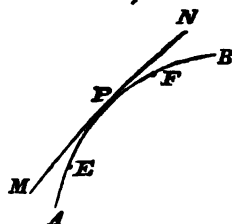
Cor. 2. Its direction at any other points, E and F, is not in the line PN, but in tangents drawn to the curve at E and F.

(53.) If the point do not continually change its direction, but move, by a sudden change at P, from the curve AP, to describe PB, there may be two tangents at the point P, PM and PN. This is not a curve of "continued curvature," but is, as it were, broken at P. Newton's expression, " Curvatura continua," (Sect. 1. Lemma 6.) is applicable to those curves alone, which are described by the motion of a point continually changing its direction.
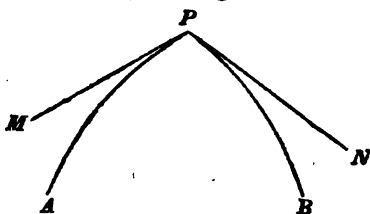
(54.) To find the length of a curve, which is referred to an axis.

Suppose AP the curve, AM the axis ; bx, PM, cv, three equidistant ordinates. Let dPs be a tangent at P,

meeting bx and cv produced in d and s. Then the direction of the point P, which describes the curve, when the ordinate is in the position MP, is in the tangent Ps. Suppose the ordinate PM, and the abscissa AM, to flow uniformly for a small space, and their fluxions to be Mc and rs; then, by Art. 44, Ps is the fluxion of AP. Let $AP = z$, $AM = x$, $MP = y$, $Pr = \dot{x}$, $rs = \dot{y}$; then $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$.
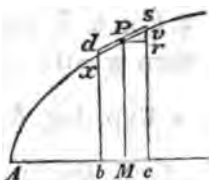
EXAMPLES.

(55.) Ex. 1. To find the length of a semicubical parabola.

The equation is $ax^2 = y^3$; $\therefore x = \dfrac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, and $\dot{x} = \dfrac{3y^{\frac{1}{2}}\dot{y}}{2a^{\frac{1}{2}}}$;

$\therefore \dot{x}^2 + \dot{y}^2 = \dfrac{9y\dot{y}^2}{4a} + \dot{y}^2 = \dfrac{(9y + 4a)\,\dot{y}^2}{4a}$; hence $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$

$= \dfrac{(9y + 4a)^{\frac{1}{2}}}{2a^{\frac{1}{2}}} \times \dot{y}$; the fluent of which, by Art. 39, $=$

$\dfrac{(9y + 4a)^{\frac{3}{2}}}{27a^{\frac{1}{2}}} + \text{corr.}$ Let the arc $= 0$; then $y = 0$;

$\therefore 0 = \dfrac{(4a)^{\frac{3}{2}}}{27a^{\frac{1}{2}}} + C$; $\therefore C = 0 - \dfrac{(4a)^{\frac{3}{2}}}{27\,a^{\frac{1}{2}}} = \dfrac{-8a^{\frac{3}{2}}}{27a^{\frac{1}{2}}} = -\dfrac{8a}{27}$;

$\therefore$ the whole length, thus corrected, $= \dfrac{(9y + 4a)^{\frac{3}{2}}}{27a^{\frac{1}{2}}} - \dfrac{8a}{27}$.

Ex. 2. To find the length of a common parabola.

Here $ax = y^2$, and $\dot{x} = \dfrac{2y\dot{y}}{a}$; $\therefore \dot{x}^2 + \dot{y}^2 = \dfrac{4y^2\dot{y}^2}{a^2} + \dot{y}^2 =$

$\left(\dfrac{4y^2}{a^2} + 1\right) \times \dot{y}^2$. Let $\dfrac{2}{a} = \dfrac{1}{b}$; then $\dfrac{4}{a^2} = \dfrac{1}{b^2}$; $\therefore \dot{z}^2 =$

$\left(\dfrac{y^2}{b^2} + 1\right)\dot{y}^2 = \dfrac{y^2 + b^2}{b^2} \times \dot{y}^2$; $\therefore \dot{z} = \dfrac{\sqrt{y^2 + b^2} \times \dot{y}}{b}$, of

which the fluent, (Fluent 58.) $= \dfrac{1}{2b} \times \sqrt{y^4 + b^2y^2} +$

$\frac{1}{2} b \times$ h. l. $(y + \sqrt{y^2 + b^2})$ + corr. Let the curve $= 0$, then $y = 0$; $\therefore 0 = \frac{1}{2} b \times$ hyp. log. $b + C$, and $C = -\frac{1}{2} b$ $\times$ hyp. log. b; $\therefore$ the whole corrected length $= \frac{1}{2b} \times$ $\sqrt{y^4 + b^2 y^2} + \frac{1}{2} b \times$ hyp. log. $(y + \sqrt{y^2 + b^2}) - \frac{1}{2} b \times$ hyp. log. $b = \frac{1}{2b} \times \sqrt{y^4 + b^2 y^2} + \frac{1}{2} b \times$ hyp. log. $(y + \sqrt{y^2 + b^2} -$ hyp. log. $b) = \frac{1}{2b} \times \sqrt{y^4 + b^2 y^2} + \frac{1}{2} b \times$ hyp. log. $\dfrac{y + \sqrt{y^2 + b^2}}{b}$.

Instead of reasoning in all cases from the expression $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$, it is sometimes useful to adopt other methods, according to the nature of the curve. Some instances are given in the following Examples.

Ex. 3. To find the length of the cycloid $ABC.$ Take $BD = a$, $BE =$ $x, BG = z, GH$ in the direction of a tangent at $G = \dot{z}$; draw HIK parallel to GE, and let $EK = \dot{x}$. Then, by similar triangles, $BEF, GIH, BE : BF :: GI : GH$; or, since $BF = \sqrt{DB \times BE} = \sqrt{ax}$, $x : a^{\frac{1}{2}} x^{\frac{1}{2}} :: \dot{x} : \dot{z}$; $\therefore \dot{z} = \dfrac{a^{\frac{1}{2}} \dot{x}}{x^{\frac{1}{2}}} = a^{\frac{1}{2}} x^{-\frac{1}{2}} \dot{x}$, and $z = 2a^{\frac{1}{2}} x^{\frac{1}{2}} +$ correction. Let $x = 0$, $z = 0$, $\therefore C = 0$; and $z = 2a^{\frac{1}{2}} x^{\frac{1}{2}}$; that is, $BG = 2BF$; $\therefore BA = 2BD$, and the whole arc of the cycloid $ABC = 4BD$.

Ex. 4. To find the length of a circular arc, by summing the four first terms of the series, which expresses the length of the arc in terms of the tangent.

Here, according to the notation in Art. 44, $\dot{z} =$

$\dfrac{a^2 \dot{t}}{a^2 + t^2}$; or, if $a^2\dot{t}$ be actually divided by $a^2 + t^2$, $\dot{z} = \dot{t} -$

$\dfrac{t^2 \dot{t}}{a^2} + \dfrac{t^4 \dot{t}}{a^4} - \dfrac{t^6 \dot{t}}{a^6} + \&c.$; $\therefore z = t - \dfrac{t^3}{3a^2} + \dfrac{t^5}{5a^4} - \dfrac{t^7}{7a^6} + \&c.$,

which needs no correction. Let the arc be $45°$, then

$t = a$; $\therefore$ an arc of $45° = a - \dfrac{a}{3} + \dfrac{a}{5} - \dfrac{a}{7} + \&c.$; or, if we

take four terms, the arc $= a \left(1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7}\right) = \dfrac{76a}{105}$;

and, if for a we write $\dfrac{d}{2}$, d being the diameter, the arc of

$45° = \dfrac{38d}{105}$; $\therefore$ the whole circumference $= \dfrac{304d}{105}$; or,

the circumference : the diameter :: 3 : 1, nearly.

Cor. 1. If we assume z an arc of $30°$, and $a = 1$, then

$t = \dfrac{1}{\sqrt{3}} = 0.5773502$; if this value be substituted for t,

and twelve terms of the series be taken, z the arc of $30°$
$= 0.5235987$. Multiply this by 12; then the whole
circumference $= 6.2831804$, where radius is 1; but the
circumferences of circles are as their radii; $\therefore$ if the
diameter be assumed $= 1$, or radius $= \frac{1}{2}$, the whole cir-
cumference in this case $= 3.141590$, &c.

Cor. 2. Since the area of a circle $= \dfrac{\text{rad.} \times \text{circumf.}}{2}$,

$\therefore$ the area of a circle, whose radius is 1, $= \dfrac{1 \times 6.2831804}{2}$

$= 3.14159$, &c.

Ex. 5. Compare the circumference of a circle with
its diameter, by summing the four first terms of a series
expressing the length of a circular arc in terms of the
sine.

Here, according to the notation, (Art. 44.) $\dot{z} =$

$\dfrac{a\dot{y}}{\sqrt{a^2-y^2}}$, of which the fluent $= y + \dfrac{y^3}{2 \cdot 3a^2} + \dfrac{3y^5}{2 \cdot 4 \cdot 5a^4} +$

$\dfrac{15y^7}{2 \cdot 4 \cdot 6 \cdot 7a^6} + $ &c. (Fluent 93.)   Let $y = a$, the arc

$=$ a quadrant; $\therefore$ the arc of a quadrant $= a \times$

$: 1 + \dfrac{1}{6} + \dfrac{3}{40} + \dfrac{15}{336}$; or, by adding the three first terms

together, and dividing the numerator and denominator

of the first by 2, and of the last by 3, it $= a \times : \dfrac{149}{120} +$

$\dfrac{5}{112} = a \times \dfrac{2161}{1680}$; $\therefore$ the whole circumference $= a \times$

$\dfrac{8644}{1680}$; or, if $a = \dfrac{d}{2}$, d being the diameter, it $= \dfrac{d \times 4322}{1680}$;

$\therefore$ the circumference : the diameter :: 4322 : 1680
nearly. If a greater number of terms were taken, the
approximation would be more correct.

(56.) Nearly in the same manner, an approximation
may be made to the value of the circumference in terms
of the diameter, by taking the sum of a few of the first
terms of the several series which express the length of
a circular arc, in terms of the cosine, versed sine, or
secant. The expression for $\dot{z}$ in terms of the cosine is

$\dot{z} = \dfrac{-a\dot{x}}{\sqrt{a^2-x^2}} = -\dot{x}\left(1 - \dfrac{x^2}{a^2}\right)^{-\frac{1}{2}}$; for the versed sine, $\dot{z} =$

$\dfrac{a\dot{x}}{\sqrt{2ax-x^2}} = \dfrac{a\dot{x}}{\sqrt{2ax}\left(1 - \dfrac{x^2}{2ax}\right)^{\frac{1}{2}}} = \dfrac{a^{\frac{1}{2}}\dot{x}}{\sqrt{2x}}\left(1 - \dfrac{x^2}{2ax}\right)^{-\frac{1}{2}}$;

and for the secant $\dot{z} = \dfrac{a^2\dot{s}}{s \times \sqrt{s^2-a^2}} = \dfrac{a^2\dot{s}}{s^2}\left(1 - \dfrac{a^2}{s^2}\right)^{-\frac{1}{2}}$.

The part, which is in the form of a residual, is to be
expanded by the binomial theorem, as in the last case,
and the fluent of each term to be taken separately.

Chap. IX.

TO FIND THE SURFACES OF SOLIDS.

(57.) Let *BAC* represent a solid
generated by the circle *PF*, as in
Art. 50. The surface of the solid is
generated by the circumference of that
circle. Hence the fluxion of the sur-
face will equal the circumference of
PF multiplied by the velocity of the
point *P*, or equal the circumference *PF*
multiplied by the fluxion of *AP*. Let
$PM = y$, $AP = x$, $p = 3.14159$, &c. the circumference of
a circle whose diameter is 1; then the fluxion of the
surface $= 2py\dot{x}$.
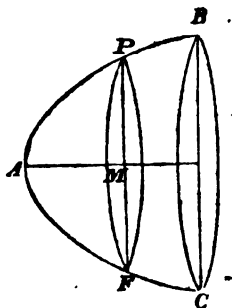
Examples.

(58.) Ex. 1. To find the surface of a sphere *ABP*.
Let *EP*, *FD* be two ordinates,
PD a tangent at *P*, *PH* perpen-
dicular to *FD*. Then if $CP = a$,
$AE = x$, $AP = x$, $EP = y$, $PH = \dot{x}$, $PD = \dot{x}$, by similar triangles
PHD, *CPE*, $\dot{x} : \dot{x} :: a : y$;
∴ $2py\dot{x}$, the fluxion of the surface
$= 2pa\dot{x}$; and the fluent $= 2pax +$
correction. Let $x = 0$, the surface
$= 0$, and the correction $= 0$; ∴ the surface $= 2pax$.
Assume $x = 2a$; then the whole surface $= 4pa^2 =$ the area
of four great circles.
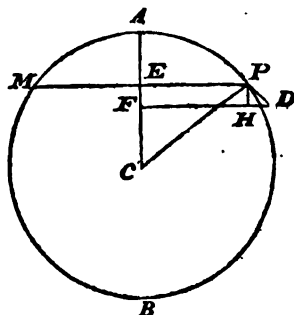
Cor. The surface $MAP \propto x$.

Ex. 2. To find the surface of the solid generated by the revolution of the common parabola about its axis.

Here $ax = y^2$; $\therefore \dot{x} = \dfrac{2y\dot{y}}{a}$, and $\dot{x}^2 = \dfrac{4y^2\dot{y}^2}{a^2}$; $\therefore \dot{z}^2 =$

$\dfrac{4y^2\dot{y}^2}{a^2} + \dot{y}^2 = \dfrac{(4y^2 + a^2)\,\dot{y}^2}{a^2}$; $\therefore \dot{z} = \dot{y} \times \dfrac{\sqrt{4y^2 + a^2}}{a}$;

hence, $2py\dot{z} = \dfrac{2py\dot{y}\sqrt{4y^2 + a^2}}{a}$, and the fluent $=$

$\dfrac{2py\dot{y}\,(4y^2 + a^2)^{\frac{3}{2}}}{8y\dot{y} \times \frac{3}{2} \times a} = \dfrac{p\,(4y^2 + a^2)^{\frac{3}{2}}}{6a}$ + correction. Let the

surface $= 0$, $y = 0$; $\therefore$ the correction $= -\dfrac{pa^2}{6}$; $\therefore$ the

corrected surface $= \dfrac{p\,(4y^2 + a^2)^{\frac{3}{2}}}{6a} - \dfrac{pa^2}{6}$.

Ex. 3. To find the surface of a cone.
Let $AC = s$, $AF = x$, $FG = y$,
$CD = b$, $FH = \dot{x}$, $HI = \dot{y}$; then,
by similar triangles, FHI, ACD,

$\dot{z} : \dot{y} :: s : b$; $\therefore \dot{z} = \dfrac{s\dot{y}}{b}$; and

the fluxion of the surface, or

$2py\dot{z} = \dfrac{2ps}{b} \times y\dot{y}$. The fluent $=$

$\dfrac{2ps}{b} \times \dfrac{y^2}{2}$, and the corr. $= 0$. Let $y = b$; then the whole

surface $= \dfrac{2psb^2}{2b} = psb = 2pb \times \dfrac{s}{2} =$ the circumference of

the base $\times \frac{1}{2}$ the side AC.
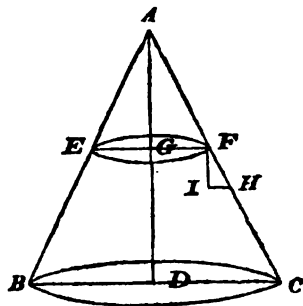
Ex. 4. To find the surface generated by the revolution of the cycloid ABC about its base DC.
Let $AD = a$, $AF = x$, $AB = z$, $BE = y$, $= a - x$; then

$AG = a^{\frac{1}{2}}x^{\frac{1}{2}}$; $AB = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$; $\therefore \dot{z} =$
$a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}$; $\therefore 2py\dot{z} = 2p(a-x)a^{\frac{1}{2}}$
$x^{-\frac{1}{2}}\dot{x} = 2pa^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x} - 2pa^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$; and

the fluent $= 4pa^{\frac{1}{2}}x^{\frac{1}{2}} - \dfrac{4pa^{\frac{1}{2}}x^{\frac{3}{2}}}{3} +$ cor-

rection, and the correction $= 0$; $\therefore$ the surface generated
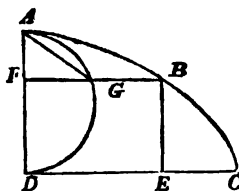
by $AB = 4pa^{\frac{1}{2}}x^{\frac{1}{2}} - \dfrac{4pa^{\frac{1}{2}}x^{\frac{3}{2}}}{3}$. Let $x = a$; then the whole

surface $= 4pa^{2} - \dfrac{4pa^{2}}{3} = \dfrac{8pa^{2}}{3}$.

Ex. 5. To find the surface of the figure called The Groin. (Vide Art. 51. Ex. 7.)

Let $AH = a$, $AF = x$, $IF = y$, $AI = z$; then, since

$\dot{z} : \dot{x} :: a : (y)\sqrt{2ax - x^{2}}$, $\dot{z} = \dfrac{a\dot{x}}{\sqrt{2ax - x^{2}}}$; also, $2FI$

$= 2 \times \sqrt{2ax - x^{2}}$; $\therefore 2py$ in the other cases $= 8 \times \sqrt{2ax - x^{2}}$ in this case; hence $2py\dot{z} = 8 \times \sqrt{2ax - x^{2}} \times$

$\dfrac{a\dot{x}}{\sqrt{2ax - x^{2}}} = 8a\dot{x}$; and the fluent $= 8ax +$ the correction,

but the correction $= 0$; $\therefore$ the surface corresponding to the abscissa $AF = 8ax$. Let $x = a$, then the whole surface $= 8a^{2}$.

CHAP. X.

ON THE CENTER OF GRAVITY.

LEMMA.

By Mechanics, if A, B, C, &c. be any number of bodies, whose center of gravity is G; and Aa, Bb, Cc, Gg, be drawn perpendicular to a plane PQ;
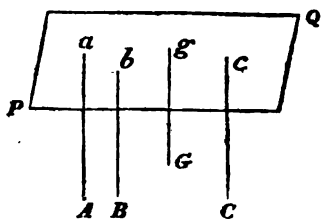
then the line $Gg = \dfrac{A \times Aa + B \times Bb + C \times Cc + \&c.}{A + B + C + \&c.}$

The truth of this Proposition is assumed in the following articles.

PROPOSITION.

(59.) To find the center of gravity of a body taken as an area, solid, curve line, or surface.

Let MAN be the body, AF its axis; draw KAL, BEC perpendicular to AF. Then, since AF is the axis, the figure will balance round it, or the center of gravity is in AF. Conceive the body to be composed of an indefinite number of particles, and multiply each particle by its distance from KL. Then, by the lemma, if G be the center of gravity, AG is equal to the sum of all these products, divided by all the particles, or by the body itself. The sum of all these products is found by taking the fluxion
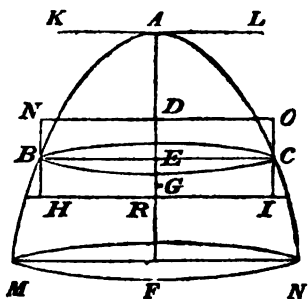
of the sum in the first instance, and then the fluent. Hence, if $\dot{s}$ represent the fluxion of the body at the distance x from KL, $AG = \dfrac{\text{the fluent of } x\dot{s}}{\text{the fluent of } \dot{s}}$.

(60.) To find the center of gravity of an area.

In this case, the fluxion of the numerator $= 2yx\dot{x}$;

$\therefore AG = \dfrac{\text{the fluent of } 2yx\dot{x}}{\text{the fluent of } 2y\dot{x}}$

$= \dfrac{\text{the fluent of } yx\dot{x}}{\text{the fluent of } y\dot{x}}$.
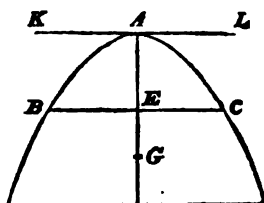
(61.) To find the center of gravity of a solid. (See the FIGURE in Art. 59.)

Here $AG = \dfrac{\text{the fluent of } py^2\dot{x} \times x}{\text{the fluent of } py^2\dot{x}} = \dfrac{\text{the fluent of } y^2 x\dot{x}}{\text{the fluent of } y^2\dot{x}}$.

(62.) To find the center of gravity of a curve line. (See the FIGURE in Art. 60.)

In this case, the fluxion of the numerator $= x\dot{z}$, and the fluxion of the denominator $= \dot{z}$; $\therefore AG = \dfrac{\text{fluent } x\dot{z}}{\text{fluent } \dot{z}}$.

(63.) To find the center of gravity of a surface.

The fluxion of the numerator $= 2pyx\dot{z}$, and that of the denominator $= 2py\dot{z}$; $\therefore AG = \dfrac{\text{fluent } 2pyx\dot{z}}{\text{fluent } 2py\dot{z}} = \dfrac{\text{fluent } yx\dot{z}}{\text{fluent } y\dot{z}}$.

EXAMPLES.

(64.) Ex. 1. To find the center of gravity of a triangle ABC.

Take AF bisecting the base; draw DE parallel, and AL perpendicular, to BC; let $AF = a$; $BC = b$, $AI = x$, $DE = y$, $AH = v$, $AL = c$. Then the center of gravity is in AF, and
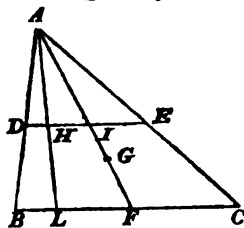
$AG = \dfrac{\text{fluent } yx\dot{x}}{\text{fluent } y\dot{x}} = \dfrac{\text{fluent } DE \times AI \times \text{the fluxion of } AH}{\text{triangle } ABC}$.

Now $AF : AL :: AI : AH$,

or, $a : c :: x : v = \dfrac{cx}{a}$; $\therefore \dot{v} = \dfrac{c\dot{x}}{a}$, and

$y = \dfrac{bx}{a}$; $\therefore AG = \dfrac{\text{fluent } \dfrac{c\dot{x}}{a} \times \dfrac{bx}{a} \times x}{\text{triangle } ABC} = \dfrac{cb \times x^3}{3a^3 \times \text{triangle}}$,

which, for the whole triangle, $= \dfrac{cb \times a^3}{3a^2 \times \dfrac{cb}{2}} = \dfrac{2a}{3}$.

Ex. 2. To find the center of gravity of the common parabola, BAC.

Here $y^2 = ax$; $\therefore y = a^{\frac{1}{2}} x^{\frac{1}{2}}$;

$\therefore \dfrac{\text{fluent } yx\dot{x}}{\text{fluent } y\dot{x}} = \dfrac{\text{fluent } a^{\frac{1}{2}} x^{\frac{3}{2}} \dot{x}}{\text{fluent } a^{\frac{1}{2}} x^{\frac{1}{2}} \dot{x}} =$

$\dfrac{\frac{2}{5} a^{\frac{1}{2}} x^{\frac{5}{2}}}{\frac{2}{5} a^{\frac{1}{2}} x^{\frac{3}{2}}}$

$\dfrac{\frac{3}{2} a^{\frac{1}{2}} x^{\frac{5}{2}}}{\frac{5}{2} a^{\frac{1}{2}} x^{\frac{3}{2}}} = \dfrac{3x}{5} = AG$.
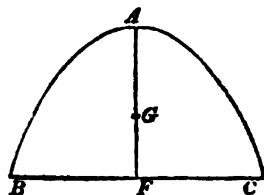
Ex. 3. To find the center of gravity of any parabola.

Here $y^n = a^{n-1} x$; $\therefore y = a^{\frac{n-1}{n}} x^{\frac{1}{n}}$; and $\dfrac{\text{fluent } yx\dot{x}}{\text{fluent } y\dot{x}} =$

$\dfrac{\text{fluent } a^{\frac{n-1}{n}} x^{\frac{1}{n}+1} \dot{x}}{\text{fluent } a^{\frac{n-1}{n}} x^{\frac{1}{n}} \dot{x}} = \dfrac{\dfrac{n+1}{n} x^{\frac{2n+1}{n}}}{\dfrac{2n+1}{n} \times x^{\frac{n+1}{n}}} = \dfrac{(n+1) x}{2n+1} = AG$.

Ex. 4. To find the center of gravity of a semi-circle.

Let FE in this case $= x$, $DE = y$, $FD = r$; then $x^2 + y^2 = r^2$;

$\therefore x\dot{x} = - y\dot{y}$; $\therefore \dfrac{\text{fluent } yx\dot{x}}{\text{fluent } y\dot{x}}$

$= \dfrac{\text{fluent} - y^2\dot{y}}{\text{fluent } y\dot{x}} = \dfrac{\text{cor.} - \frac{1}{3}y^3}{\text{area } DEFB}$.
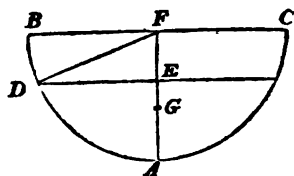
Now when the distance of the center of gravity from $BC = 0$,

DE coincides with BF; $\therefore FG = \dfrac{r^3 - y^3}{3 \times \text{area } BDEF}$; or,

for the whole semi-circle $= \dfrac{r^3}{3 \times \overline{AFB}}$.

Cor. $AG = r - \dfrac{r^3}{3 \times \overline{AFB}}$.

Ex. 5. To find the center of gravity of the segment of a circle BAC.

Take F the center; join BC, and draw FA perpendicular to BC; the center of gravity is in FA. Let $FE = x$, $ED = y$, $MB = b$; then, by the last case,
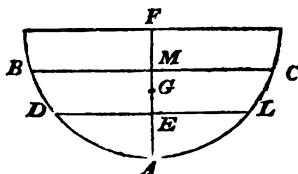
$$FG = \frac{\text{the corr.} - \frac{1}{3}y^3}{\text{the area } BMED} = \frac{b^3 - y^3}{3 \times \text{the area } BMED};$$

$\therefore FG$ for the whole segment $= \dfrac{b^3}{3 \times \text{the area } BMA}$.

Ex. 6. To find the center of gravity of the solid generated by the revolution of the common parabola about its axis.

Here $y^2 = ax$; $\therefore \dfrac{\text{fluent of } y^2 x \dot{x}}{\text{fluent of } y^2 \dot{x}}$

$= \dfrac{\text{fluent of } ax^2 \dot{x}}{\text{fluent of } ax\dot{x}} = \dfrac{\text{fluent of } x^2 \dot{x}}{\text{fluent of } x\dot{x}}$

$= \dfrac{2x^3}{3x^2} = \dfrac{2x}{3} = AG.$
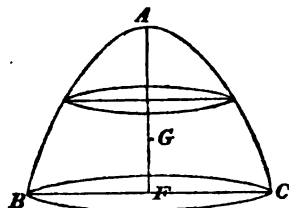
Ex. 7. To find the center of gravity of any paraboloid.

Let $y^n = a^{n-1}x$; $\therefore y = a^{\frac{n-1}{n}} x^{\frac{1}{n}}$; $\therefore y^2 = a^{\frac{2n-2}{n}} x^{\frac{2}{n}}$; and

$$\frac{\text{fluent of } y^2 x \dot{x}}{\text{fluent of } y^2 \dot{x}} = \frac{\text{fluent of } a^{\frac{2n-2}{n}} x^{\frac{2}{n}+1} \dot{x}}{\text{fluent of } a^{\frac{2n-2}{n}} x^{\frac{2}{n}} \dot{x}} = \frac{\left(\frac{2}{n}+1\right) x^{\frac{2}{n}+2}}{\left(\frac{2}{n}+2\right) \times x^{\frac{2}{n}+1}}$$

$$= \frac{(n+2)x}{2n+2}.$$

Ex. 8. To find the center of gravity of a cone.

By similar triangles, AED, AFC,

$AF : FC :: AE : ED$. Or, if

$AF = a$, and $FC = b$,

$a : b :: x : y$; $\therefore y = \dfrac{bx}{a}$, and y^2

$= \dfrac{b^2 x^2}{a^2}$; $\therefore \dfrac{\text{the fluent of } y^2 x \dot{x}}{\text{the fluent of } y^2 \dot{x}} =$

$\dfrac{\text{the fluent of } \dfrac{b^2}{a^2} x^3 \dot{x}}{\text{the fluent of } \dfrac{b^2}{a^2} x^2 \dot{x}} = \dfrac{3x^4}{4x^3} = \dfrac{3x}{4}$

$= AG.$
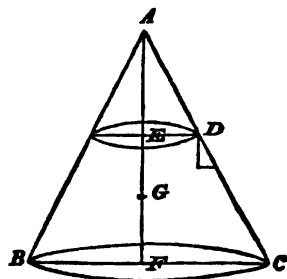
Ex. 9. To find the center of gravity of a hemisphere.
Let BAC (See the second Figure in the preceding page) represent a hemisphere; here $y^2 = 2ax - x^2$; $\therefore$ the fluent of $y^2 x \dot{x}$ = the fluent of $2ax^2\dot{x} - x^3 \dot{x} = \dfrac{2ax^3}{3} - \dfrac{x^4}{4}$; and the fluent of $y^2 \dot{x}$ = the fluent of $2ax\dot{x}$

$- x^2 \dot{x} = ax^2 - \dfrac{x^3}{3}$; $\therefore \dfrac{\text{the fluent of } y^2 x \dot{x}}{\text{the fluent of } y^2 \dot{x}} = \dfrac{\frac{2}{3}ax^3 - \frac{1}{4}x^4}{ax^2 - \frac{1}{3}x^3}$.

Let $x = AF = a$; then $AG = \dfrac{\frac{2}{3}a^4 - \frac{1}{4}a^4}{a^3 - \frac{1}{3}a^3} = \dfrac{\frac{5}{12}a^4}{\frac{2}{3}a^3} = \dfrac{5a}{8}$.

Ex. 10. To find the center of gravity of a hemispheroid.

Let BAC (See the 2d Fig. in p. 64.) represent a hemispheroid; then $y^2 = \dfrac{b^2}{a^2}(2ax - x^2)$; $\therefore$ the fluent of $y^2 x \dot{x}$

$= \dfrac{b^2}{a^2} \times$ the fluent of $(2ax^2 \dot{x} - x^3 \dot{x}) = \dfrac{b^2}{a^2} \times \left(\dfrac{2ax^3}{3}\right) - \dfrac{x^4}{4}$,

and the fluent of $y^2\dot{x} = \dfrac{b^2}{a^2} \times$ the fluent of

$(2ax\dot{x} - x^2\dot{x}) = \dfrac{b^2}{a^2}\left(ax^2 - \dfrac{x^3}{3}\right)$; $\therefore$ $\dfrac{\text{the fluent of } y^2 x\dot{x}}{\text{the fluent of } y^2\dot{x}}$

$= \dfrac{\dfrac{b^2}{a^2} \times \left(\dfrac{2}{3}ax^3 - \dfrac{1}{4}x^4\right)}{\dfrac{b^2}{a^2} \times \left(ax^2 - \dfrac{x^3}{3}\right)}$; whence by the last case, if

$x=a,\ AG = \dfrac{5a}{8}$.

Cor. 1. Since b is not found in the expression $\dfrac{5a}{8}$,
the distance of the center of gravity from A in all hemi-spheroids round the same major axis is the same.

Cor. 2. In the same manner, if BAC be an hyper-boloid, and $y^2 = \dfrac{b^2}{a^2}(2ax+x^2)$, $AG = \dfrac{8ax + 3x^2}{12a + 4x}$; that

is, if $AF = C$, $AG = \dfrac{8aC+3C^2}{12a+4C}$.

Ex. 11. To find the center of gravity of the arc of a semi-circle.

Let $FE=x$, $ED=y$, $BD=z$,
$FD=r$; then $\dot{z} : -\dot{y} :: r : x$;
(Art. 44.) $\therefore -r\dot{y} = x\dot{z}$;

$\therefore FG = \dfrac{\text{flu. of } x\dot{z}}{z} = \dfrac{\text{flu. of } -r\dot{y}}{z}$

$= \dfrac{-ry}{BD} +$ the corr.; and if $FG = 0$, $y = r$; $\therefore FG =$

$\dfrac{r^2 - ry}{BD} = \dfrac{r^2}{BDA}$ for the arc of the quadrant; and the

same is true for the other side; $\therefore FG = \dfrac{r^2}{BDA}$ for the

arc of the semi-circle.

Cor. $AG = r - \dfrac{r^2}{BDA}$.

Ex. 12. To find the center of gravity of any arc AEC of a circle.

Take S the center; and draw SDE perpendicular to the chord AC; the center of gravity G is in that line. Let $AD = b$, $SE = r$, $SB = x$, $BM = y$, $AM = z$. Then $x\dot{z} = -r\dot{y}$; $\therefore$ the fluent of $x\dot{z} = -ry +$ the corr. $= rb - ry$; and

$$SG = \frac{\text{the fluent of } x\dot{z}}{z} = \frac{rb - ry}{AM}; \text{ or}$$
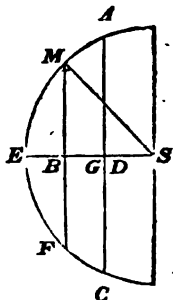
taking the whole arc AE, and therefore $y = 0$, $SG = \dfrac{r \times b}{AE} = \dfrac{SE \times AD}{AE}$; and the same conclusion holds for

EC; $\therefore SG$ for the whole arc $AEC = \dfrac{SE \times AD}{AE} = \dfrac{SE \times AC}{AEC}$.

Ex. 13. To find the center of gravity of the sector of a circle.

Let S be the center of the circle; Mm the chord of the arc MAm; SA perpendicular to Mm; QDq an arc at any distance SD; Qq its chord. Take $MAm = c$, $SM = r$, $Mm = a$, $SQ = x$, $QDq = v$; then $r : x :: a : Qq = \dfrac{ax}{r}$;
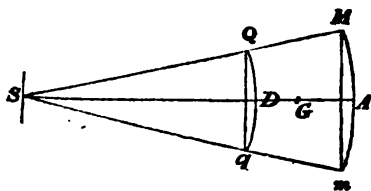
$\therefore y x \dot{x}$, in the preceding cases, corresponds in this instance with $\dfrac{ax}{r} \times x\dot{x} = \dfrac{ax^2\dot{x}}{r}$; $\therefore SG = \dfrac{\text{fluent of } y x \dot{x}}{\text{fluent of } y\dot{x}}$

$$= \frac{2}{r} \times \frac{\text{fluent of } ax^2\dot{x}}{vx} = \frac{2ax^2}{3rv}; \text{ and for the whole}$$

sector $= \dfrac{2ar}{3c}$.

Ex. 14. To find the center of gravity of the surface of a hemisphere.

Let $AE = x$, $ED = y$,

       $AD = z$, $DF = r$;

then $AG = \dfrac{\text{the fluent of } yx\dot{z}}{\text{the fluent of } y\dot{z}}$:

Now $\dot{z} : \dot{x} :: r : y$; $\therefore y\dot{z} = r\dot{x}$;

$\therefore AG = \dfrac{\text{the fluent of } rx\dot{x}}{\text{the fluent of } r\dot{x}} =$
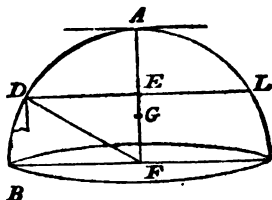
$\dfrac{x^2}{2x} = \dfrac{x}{2}$; or for the whole hemisphere $= \dfrac{r}{2}$.

Ex. 15. To find the center of gravity of the surface of a cone.

Let $AC = s$, $AF = a$, $FC = b$; (See the Fi-

gure in p. 68.) then $\dot{z} : \dot{x} :: s : a$; $\therefore \dot{z} = \dfrac{s\dot{x}}{a}$; also

$y : x :: b : a$; $\therefore y = \dfrac{bx}{a}$, and $AG = \dfrac{\text{fluent of } yx\dot{z}}{\text{fluent of } y\dot{z}}$

$$= \frac{\frac{s}{a} \times \frac{b}{a} \times \text{fluent of } x^2\dot{x}}{\frac{s}{a} \times \frac{b}{a} \times \text{fluent of } x\dot{x}} = \frac{2x}{3}; \therefore \text{ for the whole conical}$$

surface $AG = \dfrac{2AF}{3}$.

CHAP. XI.

ON THE CENTERS OF GYRATION, OSCILLATION, AND PERCUSSION.

PROPOSITION.

(65.) IF P represent any moving force acting at a distance SD from the axis S, and a body B revolve round the axis by the action of P, the same angular velocity will be produced in a given time, if a weight $= \dfrac{B \times SB^2}{SD^2}$ be placed in D.

Let M and m represent two moving forces acting at B and at D; V and v the velocities; and B and x the quantities of matter at these points. Then since $M \propto Q \times V$, we have $\dfrac{M}{m} = \dfrac{B}{x} \times \dfrac{V}{v}$; hence $\dfrac{B}{x} = \dfrac{M}{m} \times \dfrac{v}{V}$. Now since the effects are the same, we have by the lever M : m :: SD : SB; $\therefore \dfrac{M}{m} = \dfrac{SD}{SB}$; also, since the angular velocities of B and D are the same, the linear velocities are directly as the distances; or $V : v :: SB : SD$; $\therefore \dfrac{v}{V} = \dfrac{SD}{SB}$; that is, $\dfrac{B}{x} = \dfrac{SD}{SB} \times \dfrac{SD}{SB}$, or $x = \dfrac{B \times SB^2}{SD^2}$.
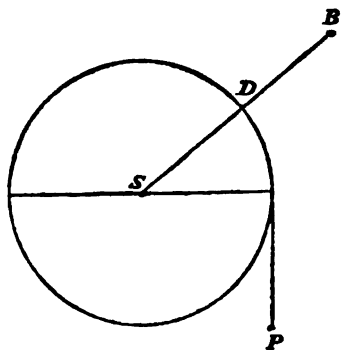
COR. Since the accelerating force $\propto$ as the moving force divided by the quantity of matter, the accelerating

force upon $D \propto$ as $\dfrac{\dfrac{P}{B \times SB^2}}{SD^2} \propto \dfrac{P \times SD^2}{B \times SB^2}$; and the same is true for any number of bodies. The inertia of P is here not taken into account.

ON THE CENTER OF GYRATION.

(66.) DEF. The center of gyration of a body, or system of bodies, is that point into which, if the whole mass were collected, a given force as P applied at a given distance from the axis of suspension, would produce the same angular velocity in the same time, as if the bodies were disposed at their respective distances.

PROPOSITION.

(67). To find the center of gyration of any body, or system of bodies, revolving round an axis of motion, which passes through S.

Let A, B, C, &c. be the bodies, or the particles of which the body is composed; P the given force applied at D; R the center of gyration. Then the force which accelerates D, whilst these bodies are at their respective distances, = (by Cor. Art. 65.)
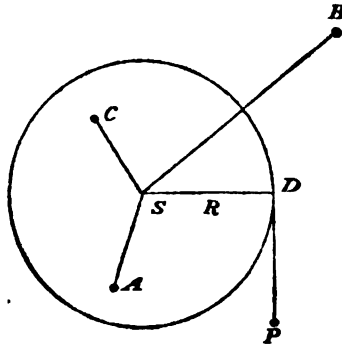
$\dfrac{P \times SD^2}{A \times SA^2 + B \times SB^2 + C \times SC^2 + \&c.}$. Next, let the whole mass be collected in R; the accelerating force upon $D = \dfrac{P \times SD^2}{(A + B + C + \&c.) \times SR^2}$. But since P, and the

angular velocity of D, are by supposition the same in both cases, the absolute velocity of D is the same, and $\therefore$ the accelerating force upon D must be the same; that is,

$$\frac{P \times SD^2}{A \times SA^2 + B \times SB^2 + \&c.} = \frac{P \times SD^2}{(A + B + C + \&c.) \times SR^2};$$

$$\therefore SR = \sqrt{\frac{A \times SA^2 + B \times SB^2 + C \times SC^2 + \&c.}{A + B + C + \&c.}}.$$

Cor. If $\dot{s}$ be the fluxion of the body at the distance x from the axis, $SR = \sqrt{\dfrac{\text{the fluent of } x^2\dot{s}}{s}}$.

ON THE CENTER OF PERCUSSION.

(68.) Def. The center of percussion is that point in the axis of a vibrating body, at which, if stopt by an immoveable obstacle, it would rest in equilibrio, without inclining to either side.

Proposition.

(69.) To find the center of percussion of any body or system of bodies.

Let RTv represent a section of the body formed by a plane passing through the center of gravity G, and perpendicular to the axis of suspension passing through S. Let O be the center of percussion; suppose the whole body to be projected orthographically upon the plane RvT, the center of gravity will remain the same, and the
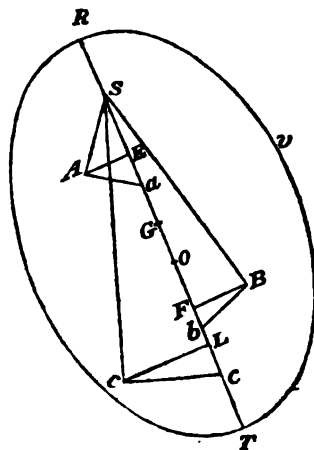
angular motion will not be affected. Let A, B, C, &c. represent particles of the body RvT; join SA, SB, SC, and draw Aa, Bb, Cc, perpendicular to these lines, meeting the axis of the body in a, b, and c; and let fall AE, BF, CL, perpendicular to ST. The instant O is stopped, the particle A will endeavour to proceed in the direction Aa, with a force proportional to $A \times SA$; and this force : its force in the direction AE, to turn the body round $O :: Aa : AE$; hence, the effect in the direction $AE = \dfrac{A \times SA \times AE}{Aa} = \dfrac{A \times SA \times SE}{SA}$ (by similar triangles) $= A \times SE$; and its effect to produce angular motion round $O = A \times SE \times aO = A \times SE \times (SO - Sa) = A \times SE \times SO - A \times SE \times Sa = A \times SE \times SO - A \times SA^2$.

In the same manner, the effect of B and C, from which the perpendiculars cut the axis in F and L, on the other side of $O = B \times SB^2 - B \times SF \times SO$; and $C \times SC^2 - C \times SL \times SO$; and since these forces balance each other round O, we have $A \times SE \times SO - A \times SA^2 = B \times SB^2 - B \times SF \times SO + C \times SC^2 - C \times SL \times SO$; $\therefore$ $(A \times SE + B \times SF + C \times SL + \&c.) \times SO = A \times SA^2 + B \times SB^2 + C \times SC^2 + \&c.$; or, by the nature of the center of gravity, $(A + B + C + \&c.) \times SG \times SO = A \times SA^2 + B \times SB^2 + C \times SC^2 + \&c.$;

$$\therefore SO = \frac{A \times SA^2 + B \times SB^2 + C \times SC^2 + \&c.}{(A + B + C + \&c.) \times SG}.$$

ON THE CENTER OF OSCILLATION.

(70.) DEF. The center of oscillation is that point in a body, or system of bodies, into which, if the whole mass were collected, it would vibrate through a given angle by the force of gravity in the same time as before.

PROPOSITION.

(71.) To find the center of oscillation.

Let *LMN* represent a body projected upon a plane, which is perpendicular to the axis of suspension passing through *S*. Let *SGO* be the axis of the body at some period of its vibration, *G* the center of gravity, and *O* the center of oscillation. Draw *SDF* perpendicular to the horizon; with *S* as a center, and *SG*, *SO*, as radii, describe the circular arcs *GD*, *OF*; draw *DE*, *FH*, perpendicular to *SO*. Let *Gv* be a line parallel to *SF*, and draw *vt* perpendicular to *SO*; join the point *S*, and the particles *A*, *B*, *C*, of the body. Now the moving force upon *G*, whilst the particles are in the position *A*, *B*, *C*, &c. : the whole weight of the body :: *tv* : *Gv* :: *DE* : *SD*; ∴ the moving force upon $G = (A + B + C + $ &c.$) \times \dfrac{DE}{SD \text{ or } SG}$; and this force will, in a given time, produce the same velocity, if *A*, *B*, *C*, &c. be removed, and the masses $\dfrac{A \times SA^2}{SG^2} + \dfrac{B \times SB^2}{SG^2} + $ &c. be placed in *G*.

Hence, the accelerating force upon *G* in this case $=$

$$\frac{(A + B + C + \text{&c.}) \times DE \times SG}{A \times SA^2 + B \times SB^2 + C \times SC^2 + \text{&c.}}.$$

Next, suppose the particles collected in *O*; the accele-

rating force on $O = \dfrac{(A+B+C+\&c.)\times\dfrac{HF}{SF}}{A+B+C+\&c.} = \dfrac{HF}{SF} = \dfrac{DE}{SG}$,
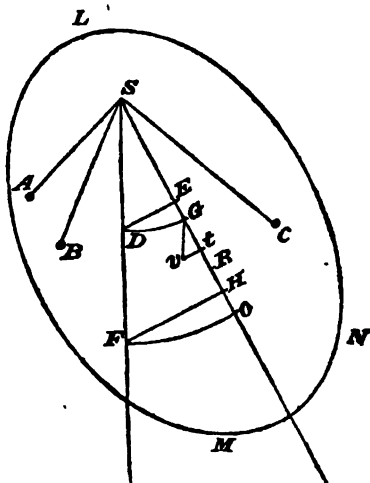
by similar triangles; and the force which accelerates O : that which accelerates G :: SO : SG; $\therefore$ the force which in this case accelerates $G = \dfrac{DE}{SO}$. But in both instances the force at G must be the same; for then, in both cases, the point G will describe a small arc, and equal successive small arcs, and consequently the whole arc GD in the same time. Hence, $\dfrac{(A+B+C+\&c.)\times DE \times SG}{A\times SA^2+B\times SB^2+C\times SC^2+\&c.}$

$= \dfrac{DE}{SO}$; $\therefore SO = \dfrac{A\times SA^2+B\times SB^2+C\times SC^2+\&c.}{(A+B+C+\&c.)\times SG}$;

so that the center of oscillation is the same point with the center of percussion.

COR. 1. If $\dot{s}$ be the fluxion of the body at the distance x from the axis, $SO = \dfrac{\text{the fluent of } x^2\dot{s}}{\text{the body} \times SG}$.

COR. 2. The distance of the center of oscillation from the axis of motion, is the same as the distance of the center of percussion.

COR. 3. $SO \times SG = \dfrac{A\times SA^2+B\times SB^2+C\times SC^2+\&c.}{A+B+C+\&c.}$.

COR. 4. If R represent the center of gyration,

$$SG : SR :: SR : SO.$$

For (Art. 67.) $SR^2 = \dfrac{A\times SA^2+B\times SB^2+C\times SC^2+\&c.}{A+B+C+\&c.}$

and $SO \times SG = $ the same quantity (by the last Cor.;) $\therefore SO \times SG = SR^2$; and $SG : SR :: SR : SO$.

PROPOSITION.

(72.) If the distance of the center of gravity and the axis of motion be increased, the distance of the center of

gravity, and the center of oscillation, will be diminished in the same proportion.

Let A, B, C, &c. be particles of the body, S the point of suspension, G the center of gravity, and O of oscillation; join SA, SB, SC, and GA, GB, GC; draw Aa, Bb, Cc, perpendicular to SO. Then,

$$SA^2 = SG^2 + GA^2 + 2SG \times Ga$$
$$SB^2 = SG^2 + GB^2 + 2SG \times Gb$$
$$SC^2 = SG^2 + GC^2 - 2SG \times Gc;$$
$$\therefore A \times SA^2 + B \times SB^2 + C \times SC^2 + \&c. = (A+B+C+ \&c.)$$
$$\times SG^2 + A \times GA^2 + B \times GB^2 + C \times GC^2 + \&c. + A \times 2SG \times$$

$Ga + B \times 2SG \times Gb - C \times 2SG \times Gc$. But, by the nature of the center of gravity, $A \times 2SG \times Ga + B \times 2SG \times Gb - C \times 2SG \times Gc = 0$; divide the whole by $(A+B+C+\&c.) \times SG$; then,

$$\frac{A \times SA^2 + B \times SB^2 + C \times SC^2 + \&c.}{(A+B+C+\&c.) \times SG} = \frac{(A+B+C+\&c.) \times SG^2}{(A+B+C+\&c.) \times SG}$$
$$+ \frac{A \times GA^2 + B \times GB^2 + C \times GC^2 + \&c.}{(A+B+C+\&c.) \times SG}; \text{ or, } SO =$$

$$SG + \frac{A \times GA^2 + B \times GB^2 + C \times GC^2 + \&c.}{(A+B+C+\&c.) \times SG}; \text{ here, } SO -$$

$$SG, \text{ or } GO = \frac{A \times GA^2 + B \times GB^2 + C \times GC^2 + \&c.}{(A+B+C+\&c.) \times SG};$$

that is, $GO \propto \dfrac{1}{SG}$, for the numerator, and the other part of the denominator in the same body, or system of bodies, is a constant quantity.

COR. 1. $SG \times GO$ is a constant quantity.

COR. 2. If O be made the point of suspension, the point S will be the center of oscillation; or the center of oscillation, and the point of suspension, are convertible.
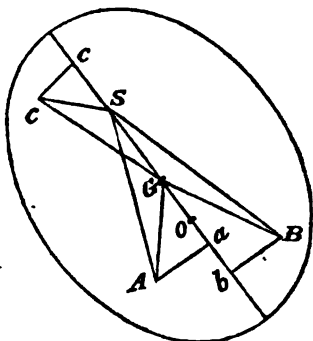
EXAMPLES.

(73.) Ex. 1. Let the straight line SA revolve about S; to find O the center of gyration.

$$SO = \sqrt{\frac{\text{the fluent of } x^2\dot{x}}{SA}} = \sqrt{\frac{SA^3}{3SA}} = \frac{SA}{\sqrt{3}},$$

when $x = SA$.

Ex. 2. To find O the center of gyration of a circle, which revolves in its own plane about its center C.

Let Ce the radius of a concentric circle $= x$, $CA = r$; then $px^2 =$ the area of the inner circle, and $2px\dot{x}$ its fluxion; $\therefore$ CO

$$= \sqrt{\frac{\text{the fluent of } 2px^3\dot{x}}{px^2}} =$$

$$\sqrt{\frac{px^4}{2px^2}} = \frac{r}{\sqrt{2}} \text{ when } x = r.$$
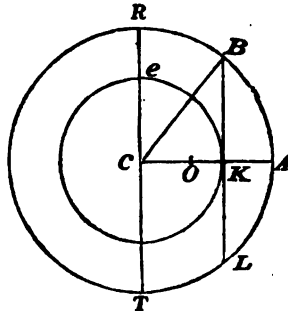
Cor. The same conclusion is true for a cylinder revolving about its axis, since it is true for every section parallel to the end.

Ex. 3. To find O the center of gyration of a sphere ABR, revolving about its diameter RT.

Draw CA perpendicular, and KB parallel to RT. Then $KB = \sqrt{r^2 - x^2}$; and the surface of the cylinder generated by the revolution of BL about $RT = 2 \times \sqrt{r^2 - x^2} \times 2px$; $\therefore$ the numerator of the expression for CO becomes $=$ the fluent of $4px^3\dot{x} \times \sqrt{r^2 - x^2}$, whose fluent, if $y = \sqrt{r^2 - x^2}$, (Fluent 75.) $= 4p \times \left(\frac{r^2y^3}{3} + \frac{1}{5}y^5\right) +$ the corr. Let $x = 0$; then $y = r$; $\therefore$ the corr. of the fluent $=$

$$4p \times \left(\frac{2}{15}r^5 - \frac{1}{3}r^2 y^3 + \frac{1}{5}y^5\right) = \text{(when } x=r, \text{ and } y = 0)$$

$$\frac{8pr^5}{15}; \text{ and the content of the sphere} = \frac{4pr^3}{3}; \therefore CO =$$

$$\sqrt{\frac{\frac{8}{15}pr^5}{\frac{4pr^3}{3}}} = r \times \sqrt{\frac{2}{5}}.$$

Case 1.

(74.) Ex. 4. Let SC be a straight line of uniform density suspended at one of its extremities S. To find the center of oscillation.

Let O be the center of oscillation, and SP a variable part $=x$. Then $SO = \dfrac{\text{the fluent of } x^2 \dot{s}}{x \times \dfrac{x}{2}} =$

$\dfrac{\text{the fluent of } x^2 \dot{x}}{\dfrac{x^2}{2}} = \dfrac{2}{3} x$, or for the whole line $= \dfrac{2SC}{3}$.

Ex. 5. To find the same, the density of the line being supposed to vary $\dfrac{1}{D^n}$ from the point of suspension.

In this case, $SO = \dfrac{\text{the fluent of } x^2 \dot{x} \times \dfrac{1}{x^n}}{\text{the fluent of } \dfrac{x\dot{x}}{x^n}} =$

$\dfrac{\text{the fluent of } x^{-n+2}\dot{x}}{\text{the fluent of } x^{-n+1}x} = \dfrac{2-n}{3-n} \times x$, for the whole line $=$

$\dfrac{2-n}{3-n} \times SC.$

Case 2.

(75.) Ex. 6. Let AB be a line vibrating lengthways in a vertical plane about S, having its two extremes A

and B equidistant from the point of suspension. To find the center of oscillation.

Draw SF perpendicular to AB; let $SF = a$, $FG = x$, and join SG. F is the center of gravity of the line; then, taking $FM = FG$, we have

$$SO = \frac{2 \times \text{fluent of } \dot{x}(a^2 + x^2)}{2x \times a}$$

$$= \frac{a^2x + \dfrac{x^3}{3}}{ax} = a + \frac{x^2}{3a} = SF +$$

$$\frac{FB^2}{3SF}.$$
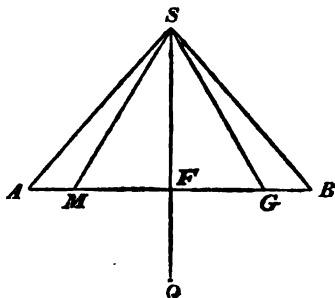
CASE. 3.

(76.) Ex. 7. Let ANM be a circle vibrating in a vertical plane.

Let a diameter MCN cut a concentric circle in D and B. Assume $SC = a$, $CB = x$, $CN = r$. Then $SD^2 + SB^2 = 2SC^2 + 2CB^2$; $\therefore$ the sum of the products of two particles at D and B, and the square of their distances from $S = \overline{a^2 + x^2} \times$ the two particles; hence that expression for the whole circumference $= \overline{a^2 + x^2} \times 2px = 2pa^2x + 2px^3$; $\therefore SO = \dfrac{\text{fluent of } x^2\dot{s}}{\text{fluent of } x\dot{s}} = \dfrac{2p \times \text{fluent of } (a^2x\dot{x} + x^3\dot{x})}{p \times ax^2}$

$$= \frac{a^2x^2 + \dfrac{x^4}{2}}{ax^2} = a + \frac{x^2}{2a} = \text{ for the whole circle } a + \frac{r^2}{2a}.$$

F
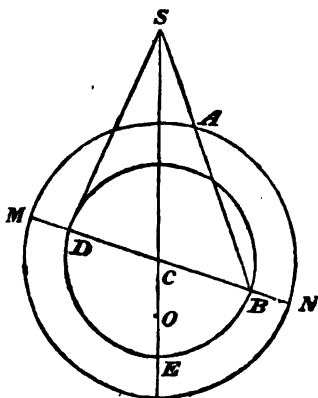
Cor. If the point of suspension S be in the circumference of this circle, $SO = r + \dfrac{r}{2}$; $\therefore CO = \dfrac{r}{2}$; and conversely, if O be made the point of suspension, where $CO = \dfrac{r}{2}$, the center of oscillation will be in the circumference. (Art. 72. Cor. 2.)

Case 4.

(77.) Ex. 8. Let ANM represent the circumference of a circle vibrating in a vertical plane.

By the last Case, the numerator of the expression for SO, or, as it is sometimes called, the force of the fluxion of the circle BDE, or of an annulus, whose breadth is $\dot{x} = 2pa^2x\dot{x} + 2px^3\dot{x}$; divide this quantity by $\dot{x}$, and there remains the force of the circumference $BDE = 2pa^2x + 2px^2$; hence, SO in this case $= \dfrac{2pa^2x + 2px^3}{\text{the circumference } 2px \times a}$

$= a + \dfrac{x^2}{a}$, or for the whole circle $= a + \dfrac{r^2}{a}$.

Case 5.

(78.) Ex. 9. Let $AHBE$ be a circle, having its plane always perpendicular to the axis of suspension SG.

The point G being the center of the circle, let AB be the diameter parallel to the axis of motion RS; draw EF parallel to AB, GP perpendicular to EF, and join SP. Take $GB = r$, $GP = x$, $SG = a$; then $EF = 2\sqrt{r^2 - x^2}$, and $SO =$

$$\frac{\text{fluent of } 2\dot{x} \times \sqrt{r^2 - x^2} \times (a^2 + x^2)}{\text{area of semi-circle} \times a}$$
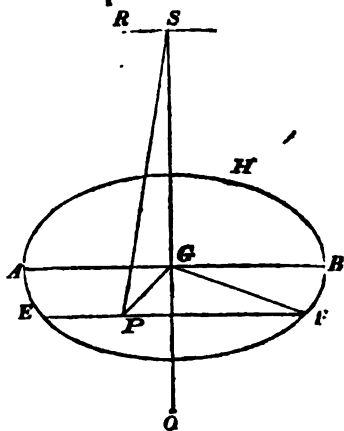
$$= \frac{\text{the fluent of } 4a^2\dot{x}\sqrt{r^2 - x^2} + 4x^2\dot{x}\sqrt{r^2 - x^2}}{\text{area of circle} \times a} =$$

Fluent 24.) $\dfrac{\left(a^2 + \frac{1}{4}r^2\right) \times pr^2}{pr^2 \times a} = a + \dfrac{r^2}{4a}$.

CASE 6.

(79.) Ex. 10. Let $AHBE$ represent the periphery of a circle, whose plane is always perpendicular to SG.

By the last Case, the numerator of the fraction, which expresses the value of SO for a circle, $= pa^2r^2 + \frac{1}{4}pr^4$; of which the fluxion $2pa^2r\dot{r} + pr^3\dot{r}$ denotes the force of an annulus whose breadth is $\dot{r}$; $\therefore$ dividing by $\dot{r}$, $2pa^2r + pr^3$ is the force of the periphery; hence, $SO = \dfrac{2pa^2r + pr^3}{2pr \times a} = a + \dfrac{r^2}{2a}$.

CASE 7.

(80.) To find the center of oscillation of a curvilinear area moving flatways, or in a direction perpendicular to its axis.

Let S be the point of suspension, $SA = d$, $AP = x$, $PN = y$; then $SO =$

the fluent of $2y\dot{x}(d + x)^2$

$$\overline{\text{fluent of } 2y\dot{x} \times \dfrac{\text{fluent of } y\dot{x}(d+x)}{\text{fluent of } y\dot{x}}}$$

$$= \frac{\text{fluent of } y\dot{x}(d+x)^2}{\text{fluent of } y\dot{x}(d+x)}.$$

If A be the point of suspension,

$$AO = \frac{\text{fluent of } yx^2\dot{x}}{\text{fluent of } yx\dot{x}}.$$
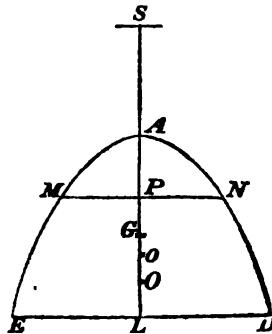
To apply these expressions to any particular curve, the value of y must be expressed in terms of x from the nature of the curve, and the fluent can be found.

Ex. 11. Let the curve be the common parabola, and S the point of suspension.

The equation is $y^2 = ax$; $\therefore SO = \dfrac{\text{fluent of } a^{\frac{1}{2}} x^{\frac{1}{2}} \dot{x} (d+x)^2}{\text{fluent of } a^{\frac{1}{2}} x^{\frac{1}{2}} \dot{x} (d+x)}$

$= \dfrac{\text{fluent of } a^{\frac{1}{2}} d^2 x^{\frac{1}{2}} \dot{x} + 2 d a^{\frac{1}{2}} x^{\frac{3}{2}} \dot{x} + a^{\frac{1}{2}} x^{\frac{5}{2}} \dot{x}}{\text{fluent of } a^{\frac{1}{2}} d x^{\frac{1}{2}} \dot{x} + a^{\frac{1}{2}} x^{\frac{3}{2}} \dot{x}} =$

$\dfrac{\frac{2}{3} d^2 x^{\frac{3}{2}} + \frac{4}{5} d \times x^{\frac{5}{2}} + \frac{2}{7} x^{\frac{7}{2}}}{\frac{2}{3} d x^{\frac{3}{2}} + \frac{2}{5} x^{\frac{5}{2}}} = \dfrac{\frac{1}{3} d^2 + \frac{2}{5} d x + \frac{1}{7} x^2}{\frac{1}{3} d + \frac{1}{5} x}.$

If $d = o$, or the point of suspension be A, $AO = \dfrac{5x}{7}.$

(81.) If A be taken for the point of suspension, the value of AO is determined by the expression $\dfrac{\text{fluent of } yx^2\dot{x}}{\text{fluent of } yx\dot{x}}$; and since AG can be found, GO may be determined. Now SG varies as $\dfrac{1}{GO}$; hence, if o be the center of oscillation, corresponding to the point of suspension S, we have $AG : SG :: \dfrac{1}{GO} : \dfrac{1}{Go}$, from which $Go = \dfrac{AG \times GO}{SG}$; and therefore So may be found.

If $SG = g$, $AG = a$, $AO = v$, and $GO = v - a$; then $Go = \dfrac{av - a^2}{g}$, and $So = g + \dfrac{av - a^2}{g} = g + \dfrac{a(v-a)}{g}.$

Ex. 12. To find the center of oscillation of the common parabola, when A is the point of suspension.

Here $AO = \dfrac{\text{fluent of } yx^2\dot{x}}{\text{fluent of } yx\dot{x}} = \dfrac{\text{fluent of } a^{\frac{1}{2}} x^{\frac{5}{2}} \dot{x}}{\text{fluent of } a^{\frac{1}{2}} x^{\frac{3}{2}} \dot{x}} =$

$\dfrac{\frac{2}{7} x^{\frac{7}{2}}}{\frac{2}{5} x^{\frac{5}{2}}} = \dfrac{5x}{7}.$

Now to determine So for some other point of suspen-

sion S; let $SA = d$, $AO = \dfrac{5x}{7} = v$, $AG = \dfrac{3x}{5} = a$

(Art. 64, Ex. 2.); $\therefore\ GO = AO - AG = \dfrac{5x}{7} - \dfrac{3x}{5}$.

Then, since $SG \times Go = AG \times GO$, or $Go =$

$\dfrac{AG \times GO}{SG} = \dfrac{\dfrac{3x}{5}\left(\dfrac{5x}{7} - \dfrac{3x}{5}\right)}{d + \dfrac{3x}{5}}$, we have $So = d +$

$\dfrac{3x}{5} + \dfrac{\dfrac{3x}{5}\left(\dfrac{5x}{7} - \dfrac{3x}{5}\right)}{d + \dfrac{3x}{5}}$, which, reduced, gives the same

result as in Ex. 11.

Ex. 13. Let EAD represent any parabola, whose equation is $a^{n-1}y = x^n$, and the point of suspension A. (See the FIGURE in p. 83.)

Here $AO = \dfrac{\text{fluent of } yx^2\dot{x}}{\text{fluent of } yx\dot{x}} = \dfrac{\text{fluent of } x^{n+2}\dot{x}}{\text{fluent of } x^{n+1}\dot{x}} =$

$\dfrac{n+2}{n+3} \times x.$

If $n = \dfrac{1}{2}$, it becomes the common parabola; and AO

$= \dfrac{5}{7} AL$ as before.

If S be the point of suspension, SO may be found

from the expression $\dfrac{\text{fluent of } y\dot{x}(d+x)^2}{\text{fluent of } y\dot{x}(d+x)}$ (Art. 80.) $=$

$\dfrac{\text{fluent of } x^n\dot{x}(d^2 + 2dx + x^2)}{\text{fluent of } x^n\dot{x}(d+x)} =$

$\dfrac{(n+2).(n+3)d^2 + (n+1).(n+3)2dx + (n+1).(n+2)x^2}{(n+2).(n+3).d + (n+1).(n+3).x};$

or by the method given in Art 81.

Ex. 14. Let AED be an isosceles triangle, suspended at A.

Take $AP = x$, $MN = y$, $AL = a$,

$ED = b$; then $AO = \dfrac{\text{fluent of } yx^2\dot{x}}{\text{fluent of } yx\dot{x}}$

$= \left(\text{since } y = \dfrac{bx}{a}\right)\dfrac{\text{fluent of } x^3\dot{x}}{\text{fluent of } x^2\dot{x}}$

$= \dfrac{3x}{4} =$ for the whole $\triangle \dfrac{3}{4} AL.$
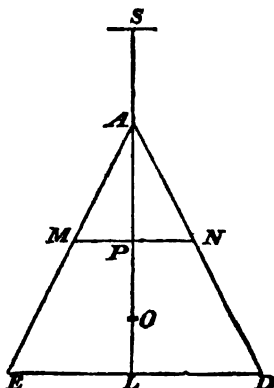
If S be the point of suspension, $So = \dfrac{\text{fluent of } (d+x)^2 \times y\dot{x}}{\text{fluent of } (d+x)\,y\dot{x}}$

$= \dfrac{6d^2 + 8dx + 3x^2}{6d + 4x}$; which, when $d = o$, becomes $\dfrac{3x}{4}$ as before.

CASE 8.

(82.) Let the proposed figure be an area EAD vibrating edgeways, so that the motion of the axis is in the plane of the curve. (See the FIGURE in the opposite page.)

The sum of the products of each particle of the line MN, and the square of its distance from S, $SP^2 \times PN$

$+ \dfrac{PN^3}{3} =$ (Ex. 6.) $(d + x)^2 \times y + \dfrac{1}{3}y^3$; $\therefore$ taking the

area AMN, $SO = \dfrac{\text{the fluent of } (d + x)^2 \times y\dot{x} + \dfrac{1}{3}y^3\dot{x}}{\text{the fluent of } (d+x)\,y\dot{x}.}$,

whence, substituting for y its value in any curve in terms of x, SO may be found.

If $d = o$, or A be the point of suspension, $AO =$

$\dfrac{\text{the fluent of } yx^2\dot{x} + \dfrac{1}{3}y^3\dot{x}}{\text{the fluent of } yx\dot{x}}$.

Ex. 15. In the common parabola, suspended at A,

$y^2 = ax$; $\therefore AO = \dfrac{\text{the fluent of } a^{\frac{1}{2}} x^{\frac{5}{2}} \dot{x} + \frac{1}{3} a^{\frac{3}{2}} x^{\frac{3}{2}} \dot{x}}{\text{the fluent of } a^{\frac{1}{2}} x^{\frac{3}{2}} \dot{x}} =$

$\dfrac{\frac{2}{7} x^{\frac{7}{2}} + \frac{2}{15} a x^{\frac{5}{2}}}{\frac{2}{5} x^{\frac{5}{2}}} = \frac{5}{7} AL + \frac{1}{3} a$, for the whole area.

Ex. 16. In any parabola, whose equation is $a^{n-1} y$

$= x^n$, $AO = \dfrac{\text{the fluent of } \dfrac{x^{n+2} \dot{x}}{a^{n-1}} + \frac{1}{3} \dfrac{x^{3n} \dot{x}}{a^{3n-3}}}{\text{the fluent of } \dfrac{x^{n+1} \dot{x}}{a^{n-1}}} =$

$\dfrac{\dfrac{x^{n+3}}{n+3} + \dfrac{1}{9n+3} \times \dfrac{x^{3n+1}}{a^{2n-2}}}{\dfrac{x^{n+2}}{n+2}} = \dfrac{n+2}{n+3} \times x + \dfrac{n+2}{9n+3} \times \dfrac{x^{2n-1}}{a^{2n-2}}.$

Ex. 17. In an isosceles triangle, $y : x :: b : a$; $\therefore AO$

$= \dfrac{\text{the fluent of } \dfrac{b}{a} x^2 \dot{x} + \frac{1}{3} \times \dfrac{b^3}{a^3} x^2 \dot{x}}{\text{the fluent of } \dfrac{b}{a} x^2 \dot{x}} = \frac{3}{4} x + \frac{1}{4} \dfrac{b^2 x}{a^2}$, when

$x = a$, and $y = b$, $= \frac{3}{4} a + \dfrac{b^2}{4a} = \dfrac{3a^2 + b^2}{4a}$. When

$SO = a$, then $\dfrac{3a^2 + b^2}{4a} = a$, and $b = a$; or the isosceles

triangle must be right-angled at the vertex.

CASE 9.

(83.) Let EAD represent a curve line vibrating flatways.

Since each particle of the line MN moves as with a radius SP, if the fluxion of $AM = \dot{x}$, $SO = \dfrac{\text{fluent of } (d + x)^2 \times \dot{x}}{\text{fluent of } (d + x) \dot{x}}$. Hence, if x can be expressed in terms of x, SO can be found.
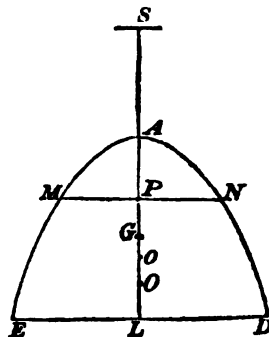
If A be the point of suspension, $d = o$, and $AO =$ $\dfrac{\text{fluent of } x^2 \dot{z}}{\text{fluent of } x \dot{z}}$.

Ex. 18. Let AED be an isosceles triangle; s one side, and a the perpendicular, the point of suspension being A. Then $\dot{z} = \dfrac{s \dot{x}}{a}$; and $AO = \dfrac{\text{the fluent of } x^2 \dot{x}}{\text{the fluent of } x \dot{x}}$ $= \dfrac{2}{3} AL.$

Ex. 19. To find the center of oscillation of any given arc of a circle MNQ, the point of suspension being in the center of the circle.

Let G be the center of gravity of the arc, MQ its chord. Then SO $= \dfrac{SN^2 \times \text{arc } MNQ}{SG \times \text{arc } MNQ} = \dfrac{SN^2}{SG} =$ $\dfrac{SN^2}{SN \times \dfrac{MQ}{MNQ}}$, (for $SG = \dfrac{SN \times MQ}{MNQ}$,
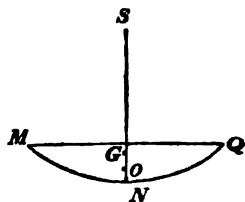
Art. 64, Ex. 12.) $= \dfrac{SN \times MNQ}{MQ}$.

CASE 10.

(84.) Let EAD be a curve line vibrating edgeways. If S be the point of suspension, the radius of the circle in which $\dot{z}$ moves $= \sqrt{(d + x)^2 + y^2}$; $\therefore SO =$ $\dfrac{\text{fluent of } \dot{z}(d + x)^2 + y^2 \dot{z}}{\text{fluent of } (d + x) \dot{z}}$.

If A be the point of suspension, $AO = \dfrac{\text{fluent of } x^2 \dot{z} + y^2 \dot{z}}{\text{fluent of } x \dot{z}}$.

Ex. 20. Let the figure be an isosceles triangle, whose altitude is a, and $b = \frac{1}{2}$ the base. Then $\dot{z} = \dfrac{s \dot{x}}{a}$; $y^2 =$

$\dfrac{b^2x^2}{a^2}$; $\therefore AO = \dfrac{\text{fluent of } x^2 \times \dfrac{s\dot{x}}{a} + \dfrac{b^2x^2}{a^2} \times \dfrac{s\dot{x}}{a}}{\text{fluent of } \dfrac{sx\dot{x}}{a}} =$

$\dfrac{\dfrac{1}{3}x^3 + \dfrac{b^2x^3}{3a^2}}{\dfrac{x^2}{2}} = (\text{if } x = a)\ \dfrac{2}{3}a + \dfrac{2b^2}{3a}.$

CASE 11.

(85.) Let the figure AEF be a solid, generated by the rotation of the surface EAF about its axis AL, having its base HH parallel to the axis of motion BC.

Take the circle MN, whose plane is parallel to HH. It appears, by Ex. 9, that the sum of the products of each particle of the circle MN, and the square of its distance from the axis, $= (SP^2 + \frac{1}{4}PN^2) \times$ circle $MN = \overline{(d+x)}^2 + \frac{1}{4}y^2)py^2$; hence, $py^2 \dot{x} (\overline{d + x}^2 + \frac{1}{4}y^2)$ is the fluxion of the sum of these products; and

$SO = \dfrac{\text{fluent of } py^2 \times (d + x)^2 \dot{x} + \frac{1}{4}py^4 \dot{x}}{\text{fluent of } py^2 \dot{x} \times (d + x)} =$

$\dfrac{\text{the fluent of } y^2 (d + x)^2 \dot{x} + \frac{1}{4}y^4 \dot{x}}{\text{the fluent of } y^2 \dot{x} (d + x)}.$

If the point of suspension be at the vertex A, $d = o$,

and $AO = \dfrac{\text{the fluent of } y^2x^2 \dot{x} + \frac{1}{4}y^4 \dot{x}}{\text{the fluent of } y^2 x\dot{x}}.$

Ex. 21. Let the solid be a cone, and S the point of suspension.
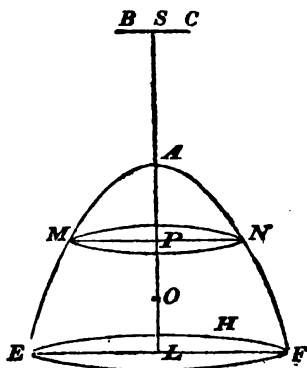
Take $AL = a$, $LF = b$; then

$$a : b :: x : y\,;\; \therefore y = \frac{bx}{a} = mx,\; \text{if}$$

$\dfrac{b}{a} = m$. Hence, the fluxion of the

numerator, which $= y^2\dot{x}\,(d+x)^2$
$+ \frac{1}{4}y^4\dot{x}$, becomes $m^2x^2\dot{x}\,(d^2 + 2dx$
$+ x^2) + \frac{1}{4}m^4x^4\dot{x}$, and the fluent $=$
$\dfrac{m^2d^2x^3}{3} + \dfrac{m^2dx^4}{2} + \dfrac{m^2x^5}{5} + \dfrac{m^4x^5}{20}.$
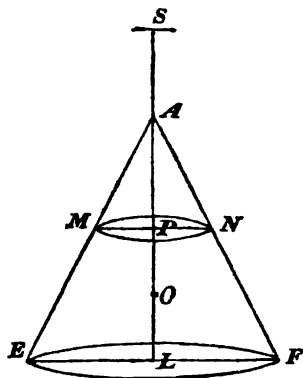
Also the fluent of the denominator $y^2\dot{x}\,(d + x) =$ the

fluent of $m^2dx^2\dot{x} + m^2x^3\dot{x} = \dfrac{m^2dx^3}{3} + \dfrac{m^2x^4}{4}\,;\;\; \therefore SO =$

$$\frac{\frac{1}{3}m^2d^2x^3 + \frac{1}{2}m^2dx^4 + \frac{1}{5}m^2x^5 + \frac{1}{20}m^4x^5}{\frac{1}{3}m^2dx^3 + \frac{1}{4}m^2x^4} =$$

$\dfrac{20d^2 + 30dx + 12x^2 + 3m^2x^2}{20d + 15x}$ by multiplying the whole

of the numerator and denominator by 60, and dividing by

m^2x^3; which for the whole cone $= \dfrac{20d^2 + 30da + 12a^2 + 3b^2}{20d + 15a}.$

COR. 1. If $d = o$, or A be the point of suspension,

$AO = \dfrac{4a^2 + b^2}{5a}.$

COR. 2. If a cone be suspended by the vertex, and the center of oscillation be in the base, $a = b$. For in this

case, $\dfrac{4a^2 + b^2}{5a} = a\,;\;\; \therefore 4a^2 + b^2 = 5a^2$, and $a = b$, a property of the right cone; so that both the right-angled cone and the right-angled isosceles triangle have their centers of oscillation in the middle of the base. See Ex. 17.

Ex. 22. Let AEF be a paraboloid, suspended at S.

Since $y^2 = ax$, SO in this case $=$

$$\frac{\text{the fluent of } axx\,(d^2 + 2dx + x^2) + \frac{1}{4} a^2 x^2 \dot{x}}{\text{fluent of } ax\dot{x} \times (d + x)} =$$

$$\frac{\frac{1}{2} ad^2 x^2 + \frac{2}{3} adx^3 + \frac{1}{4} ax^4 + \frac{1}{12} a^2 x^3}{\frac{1}{2} adx^2 + \frac{1}{3} ax^3} = \frac{6d^2 + 8dx + 3x^2 + ax}{6d + 4x}.$$

If $d = o$, $AO = \dfrac{3x + a}{4}$.

Ex. 23. Let the solid be a sphere, to find AO and So.

In this case it is most convenient to determine the value of AO, and to deduce So from it by Art. 72.

Since $y^2 = 2rx - x^2$, $AO = p$

$\times$ fluent of

$$\frac{(2rx - x^2)\, x^2 \dot{x} + \frac{1}{4}(4 r^2 x^2 - 4 r x^3 + x^4)\dot{x}}{\text{sphere} \times AG}$$

$= p \times$ fluent of

$$\frac{r^2 x^2 \dot{x} + r x^3 \dot{x} - \frac{3}{4} x^4 \dot{x}}{\text{sphere} \times AG} =$$

$$p \times \frac{\frac{1}{3} r^2 x^3 + \frac{1}{4} r x^4 - \frac{3}{20} x^5}{\frac{4pr^3}{3} \times r} = \text{(when } x = 2r) \; \frac{\frac{28}{15} r^5}{\frac{4}{3} r^4} = \frac{7r}{5}.$$
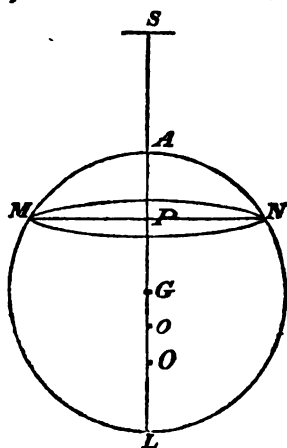
Hence, $GO = \dfrac{2r}{5}$; $\therefore$ if o be the center of oscillation, corresponding to the point of suspension S; since $SG \times Go = AG \times GO$ (Art. 72.); $\therefore (d + r)\, Go = r \times \dfrac{2r}{5} = \dfrac{2r^2}{5}$; $\therefore Go = \dfrac{2r^2}{5\,(d + r)}$; $\therefore So = d + r + \dfrac{2r^2}{5\,(d + r)}$.

If $SA = r$, $So = 2r + \dfrac{2r^2}{10r} = 2r + \dfrac{r}{5} = \dfrac{11r}{5}$.

Ex. 24. Let the solid be a cylinder, and S the point of suspension.

Here y is constant $= b$; $\therefore SO = \dfrac{b^2\dot{x}(d^2 + 2dx + x^2) + \frac{1}{4}b^4\dot{x}}{b^2\dot{x} \times (d + x)}$

$$= \frac{\text{fluent of } d^2\dot{x} + 2dx\dot{x} + x^2\dot{x} + \frac{1}{4}b^2\dot{x}}{\text{fluent of } d\dot{x} + x\dot{x}} =$$

$$\frac{d^2x + dx^2 + \frac{1}{3}x^3 + \frac{1}{4}b^2x}{dx + \frac{1}{2}x^2}.$$

Let $d = o$; $AO = \dfrac{\frac{1}{3}x^2 + \frac{1}{4}b^2}{\frac{1}{2}x} = \dfrac{2x}{3} + \dfrac{b^2}{2x}$.

CASE 12.

(86.) Let EAF be the superficies of a solid generated by the revolution of EAF round its axis; to find the center of oscillation of the superficies.

By Ex. 10. the force of the periphery of a circle $MN = 2pr$ $\left(a^2 + \frac{1}{2}r^2\right) = 2py\left(\overline{d + x}\vert^2 + \frac{1}{2}y^2\right)$; therefore the fluxion of the force of the superficies $= 2py\dot{z}$ $\left(\overline{d + x}\vert^2 + \frac{1}{2}y^2\right)$; $\therefore SO =$ $\dfrac{f \cdot 2py\dot{z}\left(\overline{d + x}\vert^2 + \frac{1}{2}y^2\right)}{\text{fluent of } 2py\dot{z}(d + x)}$.
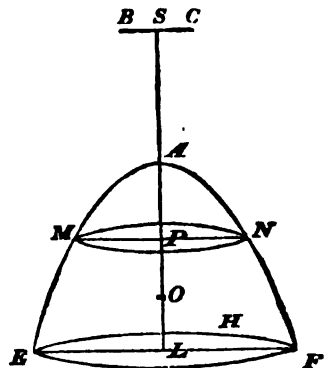
If $d = o$, $AO = \dfrac{\text{the fluent of } yx^2\dot{z} + \frac{1}{2}y^3\dot{z}}{\text{the fluent of } xy\dot{z}}$.

Ex. 25. Let EAF be a cone.

If $AE = s$, $AL = a$, $EL = b$,

$AM = z$, $\dot{z} : \dot{x} :: s : a$;

$\therefore \dot{z} = \dfrac{s\dot{x}}{a} = m\dot{x}$ if $\dfrac{s}{a} = m$; and

$x : y :: a : b$; $\therefore y = \dfrac{bx}{a} = nx$,

if $\dfrac{b}{a} = n$; hence $SO =$

$$\dfrac{f \cdot nx \times m\dot{x} \left(d^2 + 2dx + x^2 + \frac{1}{2}n^2x^2\right)}{f \cdot nx \times m\dot{x}\,(d + x)}$$

$$\dfrac{f.d^2x\dot{x} + 2dx^2\dot{x} + x^3\dot{x} + \frac{1}{2}n^2x^3\dot{x}}{f \cdot dx\dot{x} + x^2\dot{x}} = \dfrac{\frac{1}{2}d^2x^2 + \frac{2}{3}dx^3 + \frac{1}{4}x^4 + \frac{1}{8}n^2x^4}{\frac{1}{2}dx^2 + \frac{1}{3}x^3}$$

= for the whole cone, $\dfrac{12d^2 + 16da + 6a^2 + 3b^2}{12d + 8a}$.

Let $d = o$; then $AO = \dfrac{6a^2 + 3b^2}{8a}$.

Ex. 26. Let EAF be the superficies of a sphere.

Here $\dot{z} : \dot{x} :: r : y$; $\therefore y\dot{z} = r\dot{x}$. Hence, if A be the point

of suspension, $AO = \dfrac{\text{fluent of } rx^2\dot{x} + \frac{1}{2}ry^2\dot{x}}{\text{fluent of } rx\dot{x}}$; (since $y^2 =$

$2rx - x^2$) $= \dfrac{\text{fluent of } rx^2\dot{x} + r^2x\dot{x} - \frac{1}{2}rx^2\dot{x}}{\text{fluent of } rx\dot{x}} =$

$\dfrac{\text{fluent of } \frac{1}{2}rx^2\dot{x} + r^2x\dot{x}}{\text{fluent of } rx\dot{x}} = \dfrac{\frac{1}{6}rx^3 + \frac{1}{2}r^2x^2}{\dfrac{rx^2}{2}} = (\text{if } x = 2r)$

for the whole sphere $\dfrac{5r}{3}$.
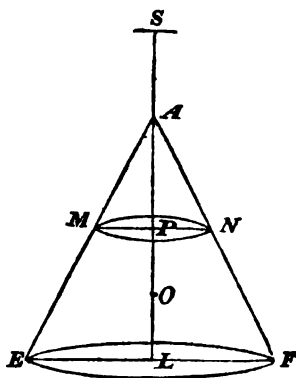
CASE 13.

(87.) Let the figure EAF be a solid, but its axis parallel to the axis of motion YRS.

Take $RP = a$, $AP = x$, $PM = y$; then, by Ex. 7, the force of the particles in the circle $MN = \left(a^2 + \frac{1}{2}y^2\right) \times py^2$; $\therefore$ SO for this solid

$$= \frac{\text{the fluent of } \left(a^2 + \frac{1}{2}y^2\right)py^2\dot{x}}{\text{solid} \times a} =$$

$$\frac{\text{the fluent of } py^2\,\dot{x} \times a^2 + \frac{1}{2}py^4\,\dot{x}}{\text{the fluent of } py^2\,\dot{x} \times a} = a +$$

$$\frac{\text{the fluent of } \frac{1}{2}y^4\,\dot{x}}{a \times \text{the fluent of } y^2\,\dot{x}}\cdot$$

Ex. 27. Let EAF be the segment of a sphere, of radius r.

Let $y^2 = 2rx - x^2$, and $y^4 = 4r^2x^2 - 4rx^3 + x^4$; $\therefore SO = a +$

$$\frac{\text{fluent of } 2r^2x^2\dot{x} - 2rx^3\dot{x} + \frac{1}{2}x^4\dot{x}}{a \times \text{fluent of } 2rx\dot{x} - x^2\dot{x}} = a + \frac{\frac{2}{3}r^2x - \frac{1}{2}rx^2 + \frac{1}{10}x^3}{\left(a \times r - \frac{1}{3}x\right)}\cdot$$

If $x = r$, then So for the hemisphere $= a + \dfrac{2r^2}{5a}$; and the same expression is true for the sphere.

Ex. 28. Let EAF be a paraboloid, whose equation is $y^2 = cx$.

$$SO = a + \frac{\text{the fluent of } \frac{1}{2}c^2x^2\dot{x}}{a \times \text{the fluent of } cx\dot{x}} = a + \frac{\frac{1}{6}c^2x^3}{\frac{1}{2}acx^2} =$$

$$a + \frac{cx}{3a} = a + \frac{y^2}{3a}\,.$$
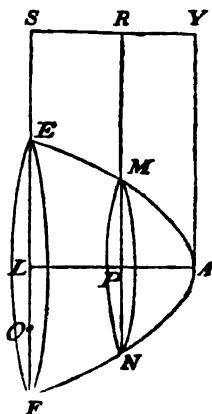
Ex. 29. Let EAF be any paraboloid, whose equation is $c^{n-1}y = x^n$.

Here $y^2 = \dfrac{x^{2n}}{c^{2n-2}}$, and $y^4 = \dfrac{x^{4n}}{c^{4n-4}}$; $\therefore SO = a +$

$$\cfrac{\displaystyle f \cdot \frac{1}{2}\frac{x^{4n}\dot{x}}{c^{4n-4}}}{\displaystyle a \times f \cdot \frac{x^{2n}\dot{x}}{c^{2n-2}}} = a + \cfrac{\displaystyle\frac{1}{8n+2}\times\frac{x^{4n+1}}{c^{4n-4}}}{\displaystyle\frac{a}{2n+1}\times\frac{x^{2n+1}}{c^{2n-2}}} = a + \frac{2n+1}{8n+2}\times\frac{x^{2n}}{ac^{2n-2}}$$

$$= a + \frac{2n+1}{(8n+2)\,a}\times y^2.$$

If $n = \dfrac{1}{2}$, it is the common parabola; and $SO = a + \dfrac{y^2}{3a}$, as before.

Ex. 30. If the figure be a cone. Let $AL = c$, $LE = b$; $y : x :: b : c$;

$\therefore y = \dfrac{bx}{c} = mx$ if $m = \dfrac{b}{c}$; $\therefore SO = a$

$+ \cfrac{\displaystyle f \cdot \frac{1}{2}m^4 x^4 \dot{x}}{\displaystyle a \times f \cdot m^2 x^2 \dot{x}} = a + \cfrac{\displaystyle\frac{1}{10}m^2 x^2}{\displaystyle\frac{1}{3}a} =$

$a + \dfrac{3}{10}\times\dfrac{b^2}{a}$ for the whole cone.
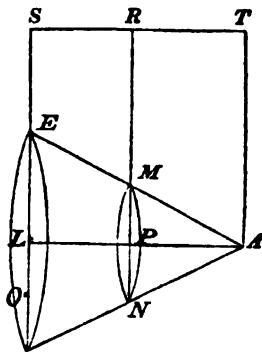
Ex. 31. In the cylinder y is constant, and $SO = a + \dfrac{y^2}{2a}$.

CASE 14.

(88.) Let EAF represent a superficies, with its axis parallel to the axis of motion. (See the FIGURE in preceding page.)

By Ex. 8. the force of the circumference of a circle vibrating in this manner, $= 2pa^2 y + 2py^3$; $\therefore$ the fluxion

of the force of this superficies $= (2pa^2y + 2py^3)\ \dot{z}$;

$$\therefore SO = \frac{f.\ 2pa^2 y\dot{z} + 2py^3\dot{z}}{a \times f.\ 2py\,\dot{z}} = a + \frac{f.\ y^3\dot{z}}{a \times f.\ y\dot{z}};\ \text{whence,}$$

by substituting for $\dot{z}$ in terms of y, SO may be found.

Ex. 32. Let EAF be a spherical superficies.

If $x =$ the versed sine, and y the ordinate, $\dot{z} = \dfrac{r\dot{x}}{y}$;

$$\therefore SO = a + \frac{f.\dfrac{y^3 \times r\dot{x}}{y}}{a \times f.\ r\dot{x}} = a + \frac{f.\ (2\,rx - x^2)\dot{x}}{a \times f.\ \dot{x}} = a + \frac{rx - \frac13 x^2}{a}\ .$$

If $x = r$, or $2r$, $SO = a + \dfrac{2r^2}{3\,a}$.

CHAP. XII.

ON SECOND, THIRD, &c. FLUXIONS.

(89.) THE fluxion of a variable quantity has been considered as its rate of increase or decrease; hence, if that increase or decrease be uniform, the fluxion continues the same. But if the rate of increase or decrease be variable, its measure will also be variable; and will itself have a certain rate of increase or decrease. The measure of this rate will be its fluxion; that is, the fluxion of the fluxion, or the second fluxion of the variable quantity. If this second fluxion be also variable, the measure of its rate of variation will be the third fluxion of the original quantity; and so on, till some fluxion becomes constant; then it will have no more. These different orders of fluxions, it is plain, are similar in their nature to the first fluxions; for they are such, in fact, to the quantities from which they are deduced; and their fluents are the fluxions which immediately precede them. The first fluxion of x being denoted by $\dot{x}$, the second fluxion is denoted by $\ddot{x}$, the third by $\dddot{x}$, &c.

EXAMPLES.

(90.) Ex. 1. The fluxion of x^2 is $2x\dot{x}$; if x increase with an uniform velocity, $\dot{x}$ is constant; but x being variable, $2x$ admits a fluxion $2\dot{x}$; and the second fluxion is $2\dot{x}^2$. But if x do not increase uniformly, $\dot{x}$ is not always the same; hence, it admits a fluxion as $\ddot{x}$; so that the second fluxion of x^2 is $2x\ddot{x} + 2\dot{x}^2$. If $\ddot{x}$ be variable, we have a third fluxion, $2\dot{x}\ddot{x} + 2x\dddot{x} + 4\dot{x}\ddot{x} = 2x\dddot{x} + 6\dot{x}\ddot{x}$. Should $\dddot{x}$

be variable, it admits a fourth, $2\dot{x}\ddot{x}+2x\dddot{x}+6\ddot{x}^2+6\dot{x}\dddot{x}$; and thus we may proceed till some one fluxion is constant.

(91.) A simple quantity x^n, where n is an affirmative whole number, has n fluxions, if $\dot{x}$ be constant.

For the first fluxion is $nx^{n-1}\dot{x}$; and $\dot{x}$ being constant;

the second - - - - - $n(n-1).x^{n-2}\dot{x}^2$;

the third - - - - - $n(n-1).(n-2)x^{n-3}\dot{x}^3$;

the nth - - - - - $n(n-1).(n-n+1)x^0\dot{x}^n$,

which is constant.

Ex. 2. To find the third fluxion of $ax^4 + by^3$.

The first fluxion $= 4ax^3\dot{x} + 3by^2\dot{y}$, where x, $\dot{x}$, y, and $\dot{y}$, are all, by hypothesis, variable; $\therefore$ the second fluxion $= 12ax^2\dot{x}^2 + 4ax^3\ddot{x} + 6by\dot{y}^2 + 3by^2\ddot{y}$; and for the same reason the third fluxion $= 24ax\dot{x}^3 + 24ax^2\dot{x}\ddot{x} + 12ax^2\dot{x}\ddot{x} + 4ax^3\dddot{x} + 6b\dot{y}^3 + 12by\dot{y}\ddot{y} + 6by\dot{y}\ddot{y} + 3by^2\dddot{y} = 24ax\dot{x}^3 + 36ax^2\dot{x}\ddot{x} + 4ax^3\dddot{x} + 6b\dot{y}^3 + 18by\dot{y}\ddot{y} + 3by^2\dddot{y}$.

Ex. 3. To find the second fluxion of x^3y^2.

The first fluxion $= 3x^2\dot{x}y^2 + 2x^3y\dot{y}$; hence, the second fluxion $= 6x\dot{x}^2y^2 + 6x^2\dot{x}y\dot{y} + 3x^2y^2\ddot{x} + 6x^2\dot{x}y\dot{y} + 2x^3\dot{y}^2 + 2x^3y\ddot{y} = 6x\dot{x}^2y^2 + 12x^2\dot{x}y\dot{y} + 3x^2y^2\ddot{x} + 2x^3\dot{y}^2 + 2x^3y\ddot{y}$.

Ex. 4. To find the second fluxion of $x^m y^n$.

The first fluxion $= mx^{m-1}\dot{x}y^n + ny^{n-1}\dot{y}x^m$; and the second fluxion $= m(m-1).x^{m-2}\dot{x}^2y^n + mx^{m-1}\ddot{x}y^n + mnx^{m-1}\dot{x}y^{n-1}\dot{y} + n(n-1)y^{n-2}\dot{y}^2x^m + ny^{n-1}\ddot{y}x^m + mny^{n-1}\dot{y}x^{m-1}\dot{x}$.

Chap. XIII.

ON THE POINT OF CONTRARY FLEXURE AND REGRESSION OF A CURVE.

Def. If a curve be in one part concave and in another convex to its axis, the point where it changes from concave to convex, or from convex to concave, is the point of contrary flexure or regression: of contrary flexure, if the curve be continued on the same side as before; of regression, if continued backward.

(92.) The abscissa being supposed to flow uniformly, it appears by Cor. Art. 30, that when the curve is concave to the axis, the fluxion of the ordinate BP decreases, or $\ddot{y}$ is negative. In curves convex to the axis, the fluxion of the ordinate increases, or $\ddot{y}$ is positive; so that at the point of contrary flexure or regression, that is, at the limiting point between the concave and the convex part the second fluxion of the ordinate changes its sign; but no quantity can change its sign, without becoming either nothing or infinite, in relation to its former magnitude; hence at the point of contrary flexure $\ddot{y} =$ nothing or infinity.

Thus, if AP be the curve, AB the abscissa, PE the curve of contrary flexure, PD that of regression, the second fluxion of the ordinate BP at the point $P = 0$, or infinity: and if AM be considered as the abscissa, and MP the ordinate, the same reasoning is true of MP.
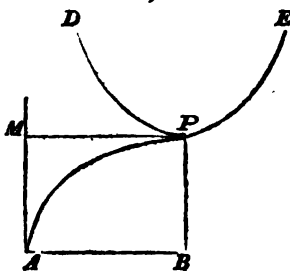
(93.) Hence the Rule. To determine the point of contrary flexure or of regression, take the fluxion of the

equation; suppose $\dot{x}$ or $\dot{y}$ constant; take the fluxion again; make $\ddot{y}$ or $\ddot{x}=0$, or infinity; and the value of x or y may be found.

And to determine, whether the part of the curve between two given points be convex or concave to the axis, observe whether the value of the expression for $\ddot{y}$ be positive or negative; if positive, the curve is convex; if negative, concave; the whole curve being supposed to be on one side of the axis.

EXAMPLES.

(94.) Ex. 1. Let the equation to the curve be $y = ax + bx^2 - cx^3$; to find the point of contrary flexure.

Here $\dot{y} = a\dot{x} + 2bx\dot{x} - 3cx^2\dot{x}$, if $\dot{x} = 1$;

$$= a + 2bx - 3cx^2;$$

$$\therefore \ddot{y} = \qquad 2b\dot{x} - 6cx\dot{x}; \text{ make } \ddot{y} = 0;$$

then, $6cx = 2b$, and $x = \dfrac{2b}{6c} = \dfrac{b}{3c} = $ the value

of x at the point of contrary flexure.

Ex. 2. To find the same in the equation $y = x + 36x^2 + 2x^3 - x^4$.

$$\dot{y} = \dot{x} + 72x\dot{x} + 6x^2\dot{x} - 4x^3\dot{x}, \text{ if } \dot{x} = 1;$$

$$= 1 + 72x + 6x^2 - 4x^3;$$

$$\therefore \ddot{y} = \qquad 72\dot{x} + 12x\dot{x} - 12x^2\dot{x}; \text{ make } \ddot{y} = 0;$$

$$\therefore 0 = 72 + 12x - 12x^2;$$

hence, $x^2 - x = 6$;

$$\therefore x^2 - x + \frac{1}{4} = \frac{25}{4}, \text{ and } x = 3, \text{ the positive value.}$$

If x be assumed less than 3 in the equation $72 + 12x - 12x^2$ which is the value of $\ddot{y}$, the result is positive; if greater than 3, the result is negative; if therefore $AB = 3$, and BC be drawn perpen-

dicular to AB, from A to C the curve is convex to the axis, and afterwards concave.

Ex. 3. Let $y = x^4 - 12x^3 + 48x^2 - 64x$, to find the point of contrary flexure.

Here $\dot{y} = 4x^3 - 36x^2 + 96x - 64$, if $\dot{x} = 1$;

$\therefore \ddot{y} = 12x^2 - 72x + 96$;

Let $\ddot{y} = 0$; then $x^2 - 6x + 8 = 0$, or $(x-2).(x-4) = 0$;

$\therefore \ddot{y} = 0$, when $x = 2$, and 4.

If x be less than 2, $\ddot{y}$ is positive, or the curve is convex to its axis; if x be greater than 2, and less than 4, $\ddot{y}$ is negative, and the curve is concave; if x be greater than 4, convex.

Ex. 4. In general, let the equation be $a^4 y = 3x^5 - 35ax^4 + 140a^2x^3 - 240a^3x^2$; $a^4\dot{y} = 15x^4 - 140ax^3 + 420a^2x^2 - 480a^3 x$, if $\dot{x} = 1$; $\therefore a^4\ddot{y} = 60x^3 - 420ax^2 + 840a^2 x - 480a^3$;

Let $\ddot{y} = 0$; then this expression $= 0$; or, dividing by 60,

$0 = x^3 - 7ax^2 + 14a^2 x - 8a^3$;

or, $0 = (x - a).(x - 2a).(x - 4a)$; therefore the curve has three points of contrary flexure corresponding to the value of x, as it $= a$, or $2a$, or $4a$.

Let x be less than a; $\ddot{y}$ is negative, or the curve concave.

If x be greater than a, but less than $2a$, $\ddot{y}$ is positive, or the curve convex.

If x be greater than $2a$, but less than $4a$, the curve is concave.

(95.) If the equation which expresses the value of $\ddot{y}$ have two equal roots, then $\ddot{y}$ does not change its sign in passing through 0; therefore the point determined by assuming $\ddot{y} = 0$, is not a point of contrary flexure. This will happen when the equation has an even number of equal roots. (Art. 27.)

(96.) To find the point of regression in a cubical para-
bola, with the equation $y = a + (a^3 - 2a^2x + ax^2)^{\frac{1}{3}}$

Here $\dot{y} = \dfrac{-2a^2 + 2ax}{3 \times (a^3 - 2a^2x + ax^2)^{\frac{2}{3}}}$,

and $\ddot{y} = - \dfrac{2a}{9 \times (a^3 - 2a^2x + ax^2)^{\frac{5}{3}}}$. From the sup-
position that $\ddot{y}=0$, no conclusion can be derived: suppose
therefore $\ddot{y} =$ infinity, it will give $9 \times (a^3 - 2a^2x + ax^2)^{\frac{5}{3}} = 0$;
or $a^2 - 2ax + x^2 = 0$, and therefore $x = a$. If this
value of x be substituted in the proposed equation, it will
give $y = a$, the value corresponding to the point of re-
gression.

CHAP. XIV.

ON THE RADIUS OF CURVATURE.

SECTION I.

PROPOSITION.

(97.) IN the common parabola, if the abscissa be per-
pendicular, and the ordinate parallel to the axis, the part
of the subsequent ordinate intercepted between the curve
and the tangent, or the deflection from the tangent, is
equal to half the second fluxion of the ordinate.

Let *AK* be the abscissa, and *ARO* the axis of the
parabola; *ak, lo, md,* three
equidistant ordinates; join
ad, and at *a* and *c,* draw tan-
gents *ab, ce*; draw *as, ct,*
parallel to *Am*; then *as, ct,*
may represent the fluxions
of the abscissa, and *sb, te,*
those of the ordinate. Now
by the nature of the parabola, *lh* bisects *ad,* and *bc = ch*;
also *oe* is parallel to *hd*; therefore *hc = de,* and consequently
bc = de; or the deflections from the tangent are uniform.
But these deflections, *bc* and *de,* are produced by ac-
celerations of velocity, which were nothing at *a* and *c*;
therefore 2 *bc* and 2 *de* will represent the spaces that
would be described by the uniform rate of increase,
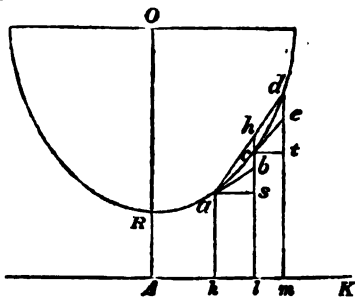
whilst bc or de is described, or they represent the fluxions of sb and te; that is, bc or $de = \frac{1}{2}$ the fluxion of $\dot{y}$, or $= \frac{1}{2} \ddot{y}$.

COR. If bc, and de, be considered as produced by accelerating forces, which were nothing at a and c, and act in the directions of the ordinates, those accelerating forces must in the parabola be uniform.

PROPOSITION.

(98.) In any algebraic curve, the deflexion from the tangent is ultimately equal to half the second fluxion of the ordinate.

In the parabola this accurately obtains, and will obtain in all curves, where the accelerating force, as in the last Corollary, can be considered as constant. By Art. 7, if the time be indefinitely diminished, the space described by a variable velocity which was nothing at first, vanishes in respect of the space described by an uniform velocity; and in the same manner the deflection from the tangent caused by a variable force, which was nothing at a, vanishes in respect of bc described by the constant force, when the time is diminished in infinitum; that is, in any algebraic curve, the deflection from the tangent ultimately equals half the second fluxion of the ordinate.

PROPOSITION.

(99.) If two algebraic curves have the same fluxions of the abscissa and ordinate, they will have the same tangent; and if the second fluxion of the ordinate be the same in both cases, they will have the same curvature.

The first part of this Proposition is evident from Art. 30, and the second is immediately deducible from the last Proposition.

DEFINITION.

Let *AB* be any curve; *BED* a circle, touching the curve in *B*; draw the ordinate *BF*; then *AF* flowing uniformly, if the second fluxion of the ordinate *BF* be the same both in the circle and the curve, *BED* is called the Circle of Curvature.
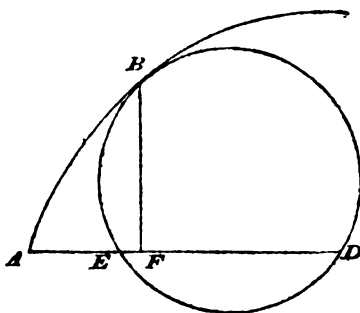
PROPOSITION.

(100.) To find the radius of curvature in any algebraic curve, in terms of the fluxions of the arc, of the ordinate, and the abscissa; the fluxion of the abscissa being constant.

Let *MAD* be the circle of curvature, touching the curve at *M*; *AP* the abscissa; *MP*, *NL*, two ordinates; *C* the center, and *MCD* the diameter of the circle; produce *MP* to *E*, and join *DE*. Let $AP = x$, $MP = y$, $MN = \dot{z}$, $SN = \dot{y}$, and $ON = \frac{1}{2}\ddot{y}$; join *MO*, *MF*. Then, by similar triangles, *NOM, NMF, NO* :
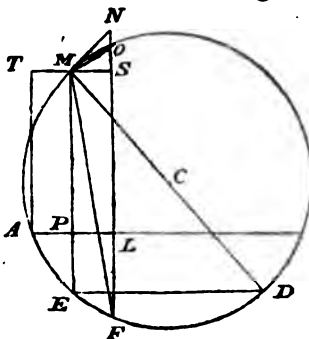
NM :: *MN* : *NF* or (Art. 97.) $- \frac{1}{2}\ddot{y} : \dot{z} :: \dot{z} : NF =$

$\dfrac{\dot{z}^2}{-\frac{1}{2}\ddot{y}}$. Now when *O* and *M* nearly coincide, *NF*

ultimately $= ME$; and, by similar triangles, *NMS, DME,*   *NM* : *MS* :: *DM* : *ME*

or $\dot{z} : \dot{x} :: DM : \dfrac{\dot{z}^2}{-\frac{1}{2}\,\dot{y}}$; $\therefore DM = \dfrac{-\dot{z}^3}{\frac{1}{2}\,\dot{x}\,\dot{y}}$; and CM

the radius of curvature $= \dfrac{-\dot{z}^3}{\dot{x}\,\dot{y}}$; $\dot{x}$ being constant.

(101.) Nearly in the same manner, by taking AT as the abscissa perpendicular to AP, and TM as the ordinate,

the radius of curvature $= \dfrac{\dot{z}^3}{\dot{y}\,\dot{x}}$; for $\dot{y}$ is uniform by sup-position, and the second fluxion of x is positive, on account of the convexity of the curve.

(102.) In general, when $\dot{x}$ and $\dot{y}$ are both variable,

the radius of curvature $= \dfrac{\dot{z}^3}{\dot{y}\,\ddot{x} - \dot{x}\,\ddot{y}}$.

Let AMN be any algebraic curve, and let MC and rC be two lines indefinitely near each other, and perpendicular to tangents at M and r. Suppose these two lines to meet in C; C is the center of the circle of curvature. Draw CF parallel to the abscissa AL, AF and MLE perpendicular; and take $AL = x$, $ML = y$, $ME = v$, $MS = \dot{x}$, $MN = \dot{z}$, $SN = \dot{y}$. Then, by similar triangles, MSN, CME, $MS : SN :: ME : EC$, or $\dot{x} : \dot{y} :: v : EC$

$= \dfrac{v\,\dot{y}}{\dot{x}}$; $\therefore FC = x + \dfrac{v\,\dot{y}}{\dot{x}}$. Now for the two radii CM,

Cr, the line FC may be considered as constant; $\therefore$ its

fluxion $= 0$; or $\dot{x} + \dfrac{(\dot{v}\dot{y} + v\ddot{y})\,\dot{x} - v\,\dot{y}\,\ddot{x}}{\dot{x}^2} = 0$; hence,

$\dot{x}^3 + \dot{v}\dot{y}\,\dot{x} + v\ddot{y}\,\dot{x} - v\,\dot{y}\,\ddot{x} = 0$; $\therefore v = \dfrac{\dot{x}^3 + \dot{v}\dot{y}\,\dot{x}}{\dot{y}\,\ddot{x} - \dot{x}\,\ddot{y}}$. But

since $MC : ME :: MN : MS$, or $MC : v :: \dot{z} : \dot{x}$, $MC =$ $\dfrac{v\dot{z}}{\dot{x}}$; hence, substituting for v its value, and $\dot{y}$ for $\dot{v}$, $MC =$

$$\frac{\dot{z}}{\dot{x}} \times \frac{\dot{x}^3 + \dot{v}\dot{y}\dot{x}}{\dot{y}\ddot{x} - \ddot{x}\ddot{y}} = \frac{\dot{x}^2\dot{z} + \dot{y}^2\dot{z}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}};$$ that is, since $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$,

$$MC = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}.$$

(103.) To find the value of the chord of curvature at M.

The diameter of curvature $= 2MC$; hence the chord $=$ $2ME$. Now, by similar triangles, MNS, CME, $MN :$ $MS :: CM : ME$; or, supposing $\dot{x}$ constant, $\dot{z} : \dot{x} :$ $\dfrac{\dot{z}^3}{-\ddot{x}\ddot{y}} : ME$; $\therefore ME = \dfrac{\dot{z}^2}{-\ddot{y}}$; and the chord, or $2ME =$ $\dfrac{2\dot{z}^2}{-\ddot{y}}$.

EXAMPLES.

(104.) **Ex. 1.** To find the radius of curvature in the common parabola.

Here $y^2 = ax$. Assume $\dot{x}$ constant and $= 1$; then since $\dot{y} = \frac{1}{2} a^{\frac{1}{2}} x^{-\frac{1}{2}}$, $\ddot{y} = \dfrac{-a^{\frac{1}{2}}}{4x^{\frac{3}{2}}}$. Again, $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} =$ $\sqrt{1 + \dfrac{a}{4x}} = \sqrt{\dfrac{4x + a}{4x}}$; $\therefore$ the radius of curvature $=$

$$\frac{\dot{z}^3}{-\ddot{x}\ddot{y}} = \left(\frac{4x + a}{4x}\right)^{\frac{3}{2}} \div \frac{a^{\frac{1}{2}}}{4x^{\frac{3}{2}}} = \frac{(4x + a)^{\frac{3}{2}}}{2\sqrt{a}}.$$

Cor. Let $x = 0$; then the radius of curvature at the vertex $= \dfrac{a}{2}$.

Ex. 2. To find the radius of curvature in any parabola, whose equation is $a^{n-1}x = y^n$.

If $\dot{y}$ be assumed constant and $= 1$, the radius of curvature $= \dfrac{\dot{z}^3}{\ddot{x}}$. Now $\dot{x} = \dfrac{ny^{n-1}}{a^{n-1}}$; $\therefore \ddot{x} = \dfrac{n(n-1)y^{n-2}}{a^{n-1}}$.

Also $\dot{x}^2 + \dot{y}^2 = \dfrac{n^2 y^{2n-2}}{a^{2n-2}} + 1 = \dfrac{n^2 y^{2n-2} + a^{2n-2}}{a^{2n-2}}$; $\therefore \dot{z}^3 =$

$\dfrac{(n^2 y^{2n-2} + a^{2n-2})^{\frac{3}{2}}}{a^{3n-3}}$; and the radius of curvature $= \dfrac{1}{a^{2n-3}}$

$\cdot \dfrac{(n^2 y^{2n-2} + a^{2n-2})^{\frac{3}{2}}}{n(n-1) y^{n-2}}$.

Cor. At the vertex $y = 0$; if $\therefore n$ be any number except 2, the radius of curvature is either nothing or infinite. If $n = 2$, the radius of curvature $= \dfrac{1}{a^2} \times \dfrac{a^3}{2.1} = \dfrac{a}{2}$.

Ex. 3. To find the radius of curvature of the common cycloid.

Let ABG be the cycloid; MC the radius of curvature at M, which is to be found; BD the axis, BOD the

generating circle, ML an ordinate parallel to the axis, and MT a line drawn parallel to the base AD. Join BO, DO. Take $AL = x$, $AM = z$, $ML = DT = y$, $DB = a$, $MN = \dot{z}$, $MS = \dot{x}$, and $NS = \dot{y}$. Suppose $\dot{x} = 1$.
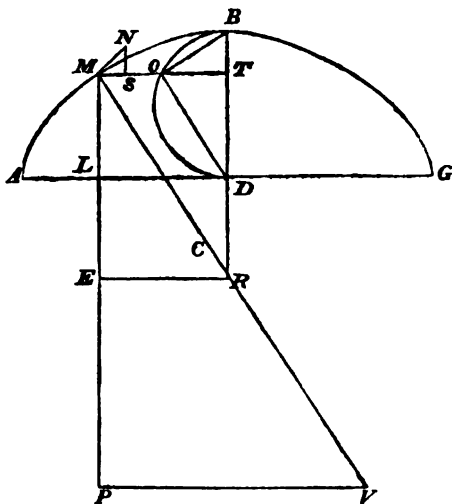
The radius of curvature in this case $= \dfrac{\dot{z}^3}{-\ddot{y}}$.

Now, by similar triangles, MSN, BOT, DTO, $\dot z$:
$\dot x :: BO : OT :: DO : DT :: \sqrt{ay} : y :: \sqrt{a} : \sqrt{y}$;

$\therefore \dot z = \left.\dfrac{a}{y}\right|^{\frac{1}{2}}$ and $\dot z^3 = \dfrac{a^{\frac{3}{2}}}{y^{\frac{3}{2}}}$. Again, by the same triangles,

$\dot x : \dot y :: OT : TB :: DT : TO :: y : \sqrt{ay - y^2} :: y^{\frac{1}{2}} :$
$\sqrt{a - y}$; $\therefore \dot y = \sqrt{\dfrac{a - y}{y}}$; and $\ddot y =$ the fluxion of $(a - y)^{\frac{1}{2}}$

$\times y^{-\frac{1}{2}} = \dfrac{-\frac{1}{2}\dot y}{y^{\frac{1}{2}}(a-y)^{\frac{1}{2}}} + \dfrac{-\frac{1}{2}\dot y(a-y)}{y^{\frac{3}{2}}(a-y)^{\frac{1}{2}}} = \dfrac{-\frac{1}{2}y\dot y - \frac{1}{2}a\dot y + \frac{1}{2}y\dot y}{y^{\frac{3}{2}}(a-y)^{\frac{1}{2}}} =$

$\dfrac{-\frac{1}{2}a\dot y}{y^{\frac{3}{2}} \times \sqrt{a-y}} = \dfrac{-\frac{1}{2}a}{y^{\frac{3}{2}} \times \sqrt{a-y}} \times \dfrac{\sqrt{a-y}}{y^{\frac{1}{2}}} = \dfrac{-a}{2y^2}$; $\therefore \dfrac{\dot z^3}{-\dot x\ddot y} =$

$\dfrac{a^{\frac{3}{2}}}{y^{\frac{3}{2}}} \times \dfrac{2y^2}{a} = 2a^{\frac{1}{2}}y^{\frac{1}{2}} = 2DO.$

Cor. Let $y = a$; then RB, the radius of curvature at the vertex, $= 2DB$.

Ex. 4. To find the chord of curvature in the cycloid at M.

Produce MR to V; take $MV = 4DO$, and draw VP perpendicular to ML produced. MP is the chord required. And by Art. 103. $MP = \dfrac{2\dot z^2}{-\ddot y} = 2 \times \dfrac{a}{y} \times \dfrac{2y^2}{a} =$

$4y = 4ML.$

SECTION II.

ON THE EVOLUTES OF CURVES.

(105.) DEF. A curve PCR is said to be the evolute of another curve AMT, when it is of such a nature that if a thread MCR wrapped round PCR were disengaged from it by some power at M, whilst the thread remained stretched, the point M would trace out the given curve AMT.
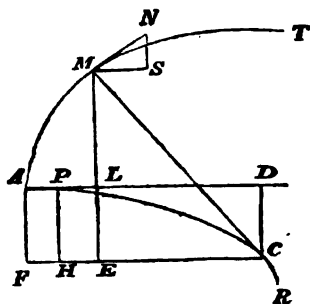
COR. 1. CM is the radius of curvature at the point M, and C is the center of curvature.

COR. 2. If A be the vertex of the curve, and AP the radius of curvature at the vertex, the evolute PCR will leave the axis APD at P; and since $APC = CM$, the curve line CP, which is the length of the evolute, $=$ $CM - AP$.

COR. 3. The same construction remaining as in Art. 102, if PH be drawn perpendicular to FC, and CD to AD, PH or CD will represent the ordinate of the curve PCR, and PD or HC the abscissa; hence, to determine the nature of the evolute, we must find PH and HC.

COR. 4. Since $PH = ME - ML$, and $HC = AD - AP$, the following lines are to be found by the nature of the curve; viz. ME, AP, and AD, or its equal $AL + LD$, or $AL + EC$.

Hence the RULE.

(106.) Having obtained the value of CM from one of the expressions for the radius of curvature, make x or

$y=0$; this will determine AP the radius of curvature at the vertex. From CM take AP; the remainder is CP the length of the evolute. Then find EL and CE from the nature of the curve, and the equation of the evolute will be determined.

<div align="center">PROPOSITION.</div>

(107.) To deduce a fluxional expression for LE and CE, in terms of the fluxion of the abscissa, of the ordinate, and of the curve.

By Art. 103, $ME = \dfrac{\dot{z}^2}{-\ddot{y}}$, if $\dot{x}$ be constant; $\therefore LE$,

which $= ME - ML, = \dfrac{\dot{z}^2}{-\ddot{y}} - y$.

Also, by similar triangles, MNS, CME,

$$MN : NS :: CM : CE \; ; \; \text{or}$$

$$\dot{z} : \dot{y} :: \dfrac{\dot{z}^2}{-\dot{x}\ddot{y}} : CE = \dfrac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}} \; ;$$

or, if $\dot{x} = 1$, $CE = \dfrac{\dot{y}\dot{z}^2}{-\ddot{y}}$.

<div align="center">EXAMPLES.</div>

(108.) Ex. 1. To find the evolute of the common parabola, and to determine its length.

Let AMT represent a parabola. Then (Art. 104. Ex. 1.)

$$CM = \dfrac{(4x+a)^{\frac{3}{2}}}{2\sqrt{a}}, \; \text{and} \; AP = \dfrac{a}{2} \; ; \; \therefore CP = \dfrac{(4x+a)^{\frac{3}{2}}}{2\sqrt{a}} - $$

$\dfrac{a}{2} =$ the length of the evolute.

Also $ME = \dfrac{\dot{z}^2}{-\ddot{y}} = \dfrac{4x+a}{4x} \times \dfrac{4x^{\frac{3}{2}}}{a^{\frac{1}{2}}}$ (Art. 104. Ex. 1.) $=$

$\dfrac{4x^{\frac{3}{2}}+ax^{\frac{1}{2}}}{\sqrt{a}}$; $\therefore LE$ or $PH = ME - ML = \dfrac{4x^{\frac{3}{2}}+ax^{\frac{1}{2}}}{\sqrt{a}} - y$

$$= \frac{4x^{\frac{3}{2}} + ax^{\frac{1}{2}}}{\sqrt{a}} - a^{\frac{1}{2}}x^{\frac{1}{2}} = \frac{4\,x^{\frac{3}{2}}}{\sqrt{a}} \;,\text{ the ordinate of the evolute.}$$

Again, $AD = AL + LD = x + EC = x + \dfrac{\dot{y}\ddot{z}^2}{-\ddot{x}\dot{y}} = x +$

$\dfrac{a^{\frac{1}{2}}}{2x^{\frac{1}{2}}} \times \dfrac{4x + a}{4x} \times \dfrac{4x^{\frac{3}{2}}}{a^{\frac{1}{2}}}$ (Art. 104.) $= x + \dfrac{4x + a}{2} = 3x + \dfrac{1}{2}\,a$;

$\therefore HC = AD - AP = 3x$, the abscissa of the evolute. Hence the square of the ordinate $\propto$ the cube of the abscissa, and the evolute is the semi-cubical parabola.

Ex. 2. To find the evolute of the common cycloid, and its length.

By Art. 104. Ex. 3, the radius of curvature $= 2DO$ in that figure; $\therefore$ at the vertex A, it $= 0$; and if R be the center of curvature at the vertex B, $RB = 2BD =$ arc AB; but $RB = RCA$; $\therefore RCA$ $= AB$; and the evolute RCA is a cycloid similar and equal to AB.
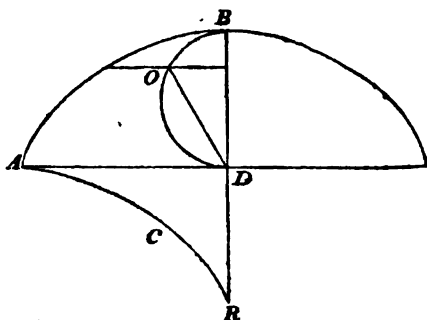
(109.) In the definition which has been given of the point of regression of a curve, it is supposed that the curve returning back towards its origin, changes from concave to convex, or from convex to concave.

But there is another species of regression; viz. when the curve in returning backwards, continues to be of the same kind with respect to concave or convex, that it was before the regression.

In illustration of this case, it may be observed, that if the value of the radius of curvature be positive, it ought to be taken on that side of the axis AL, which is supposed

in Article 102; (See the FIGURE to that Article,) and here the curve is concave to the axis. But if the radius of curvature be negative, it ought to be taken on the contrary side, and the curve is convex. Hence in the point of contrary flexure, or regression, the radius of curvature changes from + to − ; it must therefore at that point have been equal to nothing, or infinity, in relation to its former magnitude.

(110.) To find an expression for regressions of ·this second species.

Let ABK be the proposed curve, having a point of contrary flexure at B; and let an evolute be formed by a thread beginning at some point F, which is not the point of contrary flexure. The evolute of FK is FG; that of AB is IE; and that of BF is EF; so that the whole evolute of ABK is $IEFG$, which has two regressions; one at F of the common form, because the convexity at F of EF, FG is turned, and one at E of the second kind, because EF and EI are concave toward the same parts. Let ND, NCp be two rays of the evolute FB, and let NH,

CH be perpendicular to them; the infinitesimal sectors HNC, DNp are similar; and therefore $HN : ND :: NC : Dp$. But in the point B of contrary flexure the radius HB ought to be either infinite, or

to be either infinite, or equal to nothing; and the radius ND, which becomes BE, continues finite. Hence in the case of contrary flexure at B, or in the point of regression E of the second sort, the ratio of NC to Dp is infinitely great, or infinitely small; that is, the ratio of the fluxion of the
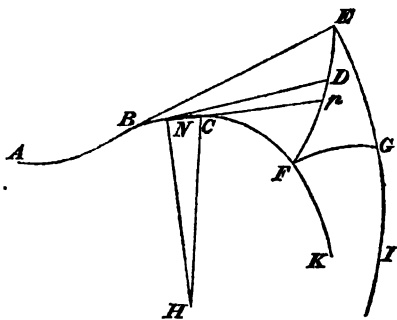
radius ND to the fluxion of the curve is infinitely great, or infinitely small; or $\dfrac{NC}{Dp} =$ infinity, or nothing. Now

$$ND = \frac{-\dot{z}^3}{\dot{x}\ddot{y}} = \frac{\overline{\dot{x}^2+\dot{y}^2}\,|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}, \text{ when } \dot{x} \text{ is constant; and its}$$

fluxion, or $NC = \dfrac{-3\dot{x}\dot{y}\ddot{y}^2\sqrt{\dot{x}^2+\dot{y}^2}+\dot{x}\ddot{y}\times\overline{\dot{x}^2+\dot{y}^2}\,|^{\frac{3}{2}}}{\dot{x}^2\ddot{y}^2}$; and

$$Dp = \dot{z} = \sqrt{\dot{x}^2+\dot{y}^2}\,; \quad \therefore \; \frac{NC}{Dp} = \frac{\dot{x}^2\dddot{y}+\dot{y}^2\dddot{y}-3\dot{y}\ddot{y}^2}{\dot{x}\ddot{y}^2} =$$

infinity, or nothing, the formula for points of regression of the second sort.

Chap. XV.

ON SPIRALS.

Definition.

If a line of inde-
finite length SA re-
volve about S, and a
point P move in it
continually from S,
the point P will trace
out . a spiral; S is
called the center, and
SP the distance.
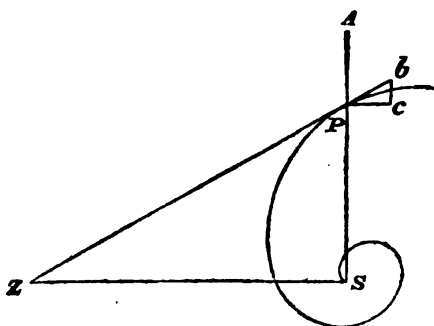
Proposition.

(111.) To draw a, tangent to a spiral at the point P.

In the revolution of SP round S, the point P has two
motions; one in a direction perpendicular to SP, and
the other in the direction of SP. Let Pc, and cb repre-
sent two small spaces, conceived to be described in these
two directions, with the velocities at P continued uniform;
join Pb. This case, therefore, coincides with that in
Art. 44; Pc represents the fluxion of the abscissa, and
cb of the ordinate; hence, Pb is the fluxion of the
curve, and Pb produced is a tangent at the point P;
draw SZ perpendicular to SP, meeting bP produced in
Z; SZ is called the sub-tangent, and its value is to be
determined in each case from the nature of the curve.

Let $SP=y$, $bc=\dot{y}$; then, by similar triangles, SPZ, Pbc, 　　$bc : cP :: SP : SZ$;

$$\text{or, } \dot{y} : cP :: y : SZ; \therefore SZ = \frac{cP \times y}{\dot{y}}.$$

PROPOSITION.

(112.) The same construction remaining,

If a circle ARx be described with a given radius $SA = a$, and if $AR = x$, and $Rx = \dot{x}$, then $SZ = \dfrac{y^2 \dot{x}}{a\dot{y}}$.
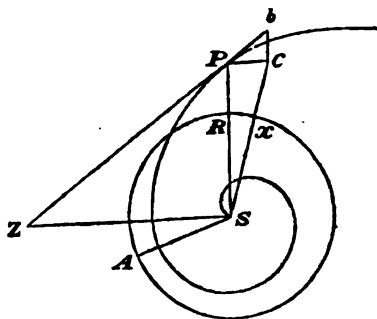
Join Sc; then, by similar triangles, Pbc, SPZ,

$$bc : cP :: SP : SZ$$
$$cP : Rx :: SP : SR$$

$$\therefore bc : Rx :: SP^2 : SR \times SZ;$$
$$\text{or, } \dot{y} : \dot{x} :: y^2 : a \times SZ;$$
$$\therefore SZ = \frac{y^2 \dot{x}}{a\dot{y}}.$$

(113.) To find the areas of spirals.

Let the spiral SfP be conceived to be described by the uniform angular motion of SP; and suppose Sf and Sb to be two distances near SP, which make equal angles with it. With S as a center, and SP as radius, describe the circular arc DPc. Then SPc being equal to SPe, is greater than the preceding increment SPf of the area, and less than the suc-

ceeding; also, since SP revolves uniformly, SPc is uniformly described; $\therefore SPc$ is the fluxion of the area SfP. Now if the arc $DP=x$, and $SP=y$, the sector $SPc = \dfrac{y\dot{x}}{2}$; $\therefore$ the fluxion of $SfP = \dfrac{y\dot{x}}{2}$.
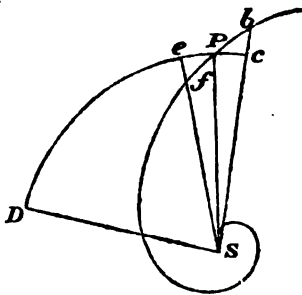
(114.) If bPy be a tangent at P, and PSb a very small angle, and Sy be drawn per-
pendicular to Py, the triangles Pcb,
SPy are ultimately similar, and $cb=\dot{y}$;
$\therefore Pc : cb :: Sy : Py$; or, if $Sy=p$,
and $Py = t$,

$$\dot{x} : \dot{y} :: p : t; \; \therefore \dot{x} = \frac{p\dot{y}}{t}.$$

And $\dfrac{y\dot{x}}{2}$, or the fluxion of the area, $=\dfrac{py\dot{y}}{2t}$.

(115.) Again, by similar triangles, Pcb, SPy,
$$Pc : Pb :: Sy : SP;$$
or, $\dot{x} : \dot{z} :: p : y$; $\; \therefore \dfrac{y\dot{x}}{2} = \dfrac{p\dot{z}}{2}$;

and the fluxion of the area $=\dfrac{p\dot{z}}{2}$.

(116.) If with S as a center, and $SA=a$ as radius, a circle be described cutting SP, SC, in R and x; and $AR=w$; the fluxion of the area $= \dfrac{y^2\dot{w}}{2a}$. (See the first FIGURE in the preceding page.)

For $Pc : Rx :: SP : SR$;
   or, $\dot{x} : \dot{w} :: y : a$;
$\therefore \dot{x} = \dfrac{y\dot{w}}{a}$; and $\dfrac{y\dot{x}}{2}$, or the fluxion of the area$= \dfrac{y^2\dot{w}}{2a}$.

Any of these expressions may be adopted to find the area.

PROPOSITION.

(117.) To find the length of spirals. (See the FIGURE in Art. 114.)
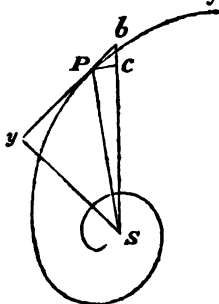
Let SP be a spiral curve described by the uniform revolution of the distance SP, whilst the point P moves continually from S. Draw Pc perpendicular to SP, and cb parallel to it; and let Pc, cb represent the uniform velocities of SP and the point P; join Pb, then Pb is in the direction of a tangent at P, and is the fluxion of the curve. Draw Sy perpendicular to the tangent; then, by similar triangles, Pbc, SPy,

$$Pb \ : \ bc \ :: \ SP \ : \ Py$$

or, $\dot{z} \ : \ \dot{y} \ :: \ y \ : \ t; \ \therefore \dot{z} = \dfrac{y\dot{y}}{t}.$

PROPOSITION.

(118.) To find the point of contrary flexure.

Let C be the point of contrary flexure; suppose the curve convex from A to C, and concave afterward toward S. Draw Sy perpendicular to the tangent Ay. Then, while A approaches C, Sy is increasing; after that time it decreases to S. Hence, if the fluxion of the perpendicular be assumed $= 0$, it will determine the point of contrary flexure.
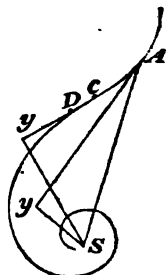
PROPOSITION.

(119.) To find an expression for the radius, and chord of curvature, in spiral curves.

Let PVM be the circle, and PCM the diameter of curvature; S the center of the spiral, and C that of the circle. Draw Sy perpendicular to the tangent Py, SR perpendicular to PM, and join SC. Let $SP = x$, $Sy = RP = p$, $CP = r$; then, by Euclid,

$$SC^2 = CP^2 + PS^2 - 2CP \times PR$$
$$= r^2 + x^2 - 2rp;$$
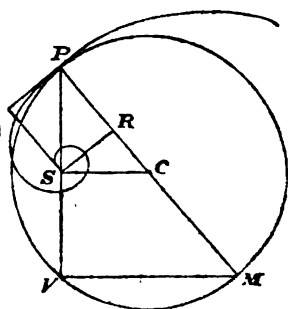
Now SC may be considered as invariable for this circle;

$\therefore$ its fluxion $=0$; hence, $0=2x\dot{x}-2r\dot{p}$, and $r=\dfrac{x\dot{x}}{\dot{p}}=CP$.

Next, to find the chord, produce PS to V; and join MV. Then, by similar triangles, SPy, MPV,
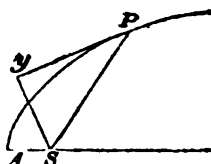
$SP : Sy :: MP : PV$;

or, $x : p :: \dfrac{2x\dot{x}}{\dot{p}} : PV=\dfrac{2p\dot{x}}{\dot{p}}=$ the chord of curvature.

EXAMPLES.

(120.)　**Ex. 1.**　To find the radius, and chord of curvature of the parabola, considered as a spiral.

Here $p^2=ax$; $\therefore 2p\dot{p}=a\dot{x}$, and $\dfrac{x\dot{x}}{\dot{p}}=$

$\dfrac{2p}{a}\times x=\dfrac{2p^3}{a^2}=$ the radius of curvature.

Also $\dfrac{\dot{x}}{\dot{p}}=\dfrac{2p}{a}$; $\therefore \dfrac{2p\dot{x}}{\dot{p}}=\dfrac{4p^2}{a}=\dfrac{4ax}{a}=4x$; or the chord $=4SP$.

Ex. 2.　Let the ellipse be considered as a spiral; to find the chord of curvature passing through the focus.

Let $AC=a$, $SP=x$, $Sy=p$;
　　　$CB=b$, $HP=v$;

then, $p^2=\dfrac{b^2\times x}{v}$, by Conic Sec-

tions; $\therefore 2p\dot{p}=\dfrac{b^2v\dot{x}-b^2x\dot{v}}{v^2}$; but

$x+v=2a$; $\therefore \dot{x}=-\dot{v}$; hence,

$2p\dot{p}=\dfrac{b^2(v+x)\dot{x}}{v^2}=\dfrac{2ab^2\dot{x}}{v^2}$; $\therefore \dfrac{\dot{x}}{\dot{p}}=\dfrac{2pv^2}{2ab^2}$, and $\dfrac{2p\dot{x}}{\dot{p}}$

$=\dfrac{4p^2v^2}{2ab^2}=\dfrac{4\times b^2xv}{2ab^2}=\dfrac{2SP\times HP}{AC}=\dfrac{2CD^2}{AC}$, where CD

is the conjugate diameter to CP.

Ex. 3.   To find the chord of curvature of the spiral, whose equation is $p = \dfrac{a\,x}{\sqrt{a^2 + x^2}}$.

Here $\dot{p} = a\dot{x}\,(a^2 + x^2)^{-\frac{1}{2}} - x\dot{x} \times a\,x\,(a^2 + x^2)^{-\frac{3}{2}} =$

$\dfrac{a^2\dot{x} + ax^2\dot{x} - a\,x^2\dot{x}}{(a^2 + x^2)^{\frac{3}{2}}} = \dfrac{a^3\,\dot{x}}{(a^2 + x^2)^{\frac{3}{2}}}$; $\therefore$ the chord of cur-

vature, or $\dfrac{2\,p\,\dot{x}}{\dot{p}} = \dfrac{2\,x\,(a^2 + x^2)}{a^2}$.

PROPOSITION.

(121.)   To determine the evolutes of spiral curves.

Let SVP represent any spiral curve ; CP the radius of curvature at the point P ; S the pole of the spiral ; Sy perpendicular to a tangent at P, and SR perpendicular to CP. Join SC, and let CT be the evolute ; then, by Art. 119, $CP = \dfrac{x\,\dot{x}}{\dot{p}}$; and CP corrected, if necessary, gives the length of the evolute. Next, find $CR = CP - PR$ the tangent of the evolute ; then, CS its ordinate, which $= \sqrt{CR^2 + RS^2}$. From these two, the nature of the curve will be known.

ON THE INVOLUTE OF A CIRCLE.

DEFINITION.

(122.)   The involute AB of a circle is described by the extremity B of a string unwinding itself from the circumference of a circle AEC.

COR. 1. A small arc at B may be considered as a circular arc, whose radius is CB; and CB is therefore perpendicular to the curve or to the tangent By.

COR. 2. If SY be drawn perpendicular to By, and SC be joined, $BCSy$ is a parallelogram; and $By = SC$, the radius of the circle.

PROBLEMS.

(123.) **PROB. 1.** To find the area of the involute.

Let $SB = y$, $By = SC = r$, $Sy = p$; then, $Sy = \sqrt{y^2 - r^2}$;

$\therefore$ the fluxion of the area, or $\dfrac{p y \dot{y}}{2t}$ (Art. 114.) $= \dfrac{\sqrt{y^2 - r^2} \times y \dot{y}}{2r}$;

and the fluent $= \dfrac{(y^2 - r^2)^{\frac{3}{2}}}{6r} +$ cor.

Let $y = SA = r$; then the area $= 0$; therefore cor. $= 0$;

$\therefore$ the corrected fluent or the area $SAB = \dfrac{(y^2 - r^2)^{\frac{3}{2}}}{6r} = \dfrac{Sy^3}{6SC}$.

COR. Since $CB = CEA$, $\dfrac{CB \times SC}{2} = \dfrac{CEA \times SC}{2}$, or the area of the triangle $SCB =$ the sector ASC; hence the area $ASCB -$ the triangle $SCB =$ the area $ASCB -$ the sector ASC; that is, the area $ASB =$ the area $AECB$; consequently the area $AECB = \dfrac{Sy^3}{6SC}$.

PROB. 2. To find the length of the involute.

Here $\dot{z} = \dfrac{y\dot{y}}{t} = \dfrac{y\dot{y}}{r}$; $\therefore z = \dfrac{y^2}{2r} + $ cor.; but when $y = r$,

$z = 0$; $\therefore$ cor. fluent $= \dfrac{y^2 - r^2}{2r} = \dfrac{Sy^2}{2SC}$.

———◆———

ON THE SPIRAL OF ARCHIMEDES.

(124.) DEF. If with S as a pole, a spiral SEP be described of such a nature, that the distance SP always bears a given ratio to the arc AB of a given circle, whose center is S, SEP is called the Spiral of Archimedes.
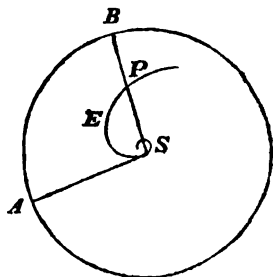
PROBLEMS.

(125.) PROB. 1. To draw a tangent to this spiral.

Let PZ represent the tangent at P, SZ the sub-tangent; SY a perpendicular on PZ. Then if $AB = x$, $SP = y$, $SA = a$, $SZ = \dfrac{y^2 \dot{x}}{a\dot{y}}$ (Art. 112.); but here $x : y :: c : d$, a given ratio; $\therefore \dot{x} = \dfrac{c\dot{y}}{d}$; $\therefore SZ =$

$\dfrac{y^2}{a\dot{y}} \times \dfrac{c\dot{y}}{d} = \dfrac{cy^2}{ad} = \dfrac{y^2}{b}$, if $\dfrac{ad}{c}$ be assumed $= b$.

PROB. 2. To find the length of the tangent PZ.

$PZ^2 = SP^2 + SZ^2 = y^2 + \dfrac{y^4}{b^2} = \dfrac{y^2(b^2 + y^2)}{b^2}$; therefore

$PZ = \dfrac{y \times \sqrt{b^2 + y^2}}{b}$.
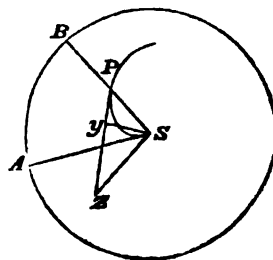
PROB. 3. To find PY.

We have $ZP : PS :: PS : PY$;

or, $\dfrac{y}{b} \times \sqrt{b^2+y^2} : y :: y : PY = \dfrac{by}{\sqrt{b^2+y^2}}$.

PROB. 4. To find the perpendicular SY.

By similar triangles, SPY, SPZ.

$$PZ : SZ :: SP : SY;$$

or, $\dfrac{y}{b} \times \sqrt{b^2+y^2} : \dfrac{y^2}{b} :: y : SY = \dfrac{y^2}{\sqrt{b^2+y^2}}$.

PROB. 5. To find its area.

One expression for the fluxion of the area is $\dfrac{y^2\dot{w}}{2a}$ (Art. 116.) where $\dot{w}$ is the fluxion of a circular arc of radius a, or in this case the fluxion of AB; ∴ here the expression is $\dfrac{y^2\dot{x}}{2a}$.

Now, $x : y :: c : d$; ∴ $\dot{x} = \dfrac{c\dot{y}}{d}$;

∴ the fluxion of the area $= \dfrac{y^2}{2a} \times \dfrac{c\dot{y}}{d} = \dfrac{cy^2\dot{y}}{2ad} = \dfrac{y^2\dot{y}}{2b}$;

∴ the fluent $= \dfrac{y^3}{6b}$; cor. $= 0$; and the area $= \dfrac{y^3}{6b}$.

PROB. 6. To find its length.

By Art. 117. $\dot{s} = \dfrac{y\dot{y}}{t}$; in this case, $t = \dfrac{by}{\sqrt{b^2+y^2}}$;

∴ $\dot{s} = \dfrac{y\dot{y} \times \sqrt{b^2+y^2}}{by} = \dfrac{\dot{y} \times \sqrt{b^2+y^2}}{b}$; whose fluent (fl. 58.)

$= \dfrac{1}{2b} \times \sqrt{y^4+b^2y^2} + \dfrac{1}{2} b \times \text{hyp. log.} \dfrac{y+\sqrt{y^2+b^2}}{b}$.

PROB. 7. To find the radius of curvature, or the length of the evolute.

By Art. 119, the radius of curvature $= \dfrac{x\dot{x}}{\dot{p}}$, where

$x = SP$; that is, in this case it $= \dfrac{y\dot{y}}{\dot{p}}$. Now $p =$

$\dfrac{y^2}{\sqrt{y^2 + b^2}}$; $\therefore \dot{p} = \dfrac{y^2\dot{y} + 2b^2 y\dot{y}}{(y^2 + b^2)^{\frac{3}{2}}}$ (Art. 20. Ex. 23.);

hence, the radius of curvature, or $\dfrac{y\dot{y}}{\dot{p}} = \dfrac{(y^2 + b^2)^{\frac{3}{2}}}{y^2 + 2b^2} + $ cor.

Let $y = 0$, this expression $= \dfrac{b}{2}$; $\therefore$ cor. $= -\dfrac{b}{2}$; that is,

the radius of curvature, or the length of the evolute

reckoned from the vertex, $= \dfrac{(y^2 + b^2)^{\frac{3}{2}}}{y^2 + 2b^2} - \dfrac{b}{2}$.

ON THE SPIRAL OF PAPPUS.

(126.) Let C be the center of the sphere; $ARaB$ a great circle; P its pole. Let the quadrant PMA revolve uniformly round the pole P, while at the same time, a point

proceeding from P moves uniformly with a given velocity along the quadrant, and traces out upon the spherical surface, the spiral $PFfa$. This spiral is analogous to that described upon a plane by Archimedes.
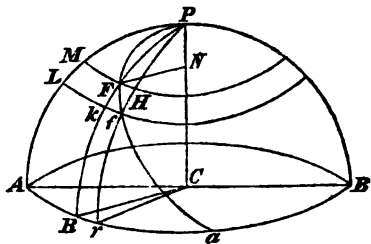
PROBLEMS.

(127.) To find the spherical surface, included between the quadrant PMA, and the spiral.

Let PFR, Pfr be two positions of the revolving quadrant very near each other, and F and f corresponding

points, in the spiral line; with P as a center, describe the parallel small circles, MFH, Lkf, and let the great circles PF, Pf meet the great circle $ARaB$ in R, r. From F in the plane of the circle PFR, draw FN perpendicular to the axis CP; then will FN be the sine, and CN the cosine of FP.

Let the circular arc $PF = z$; the radius $CA = CR = r$: $FN = y$; $CN = x$, and assume $Hf : Rr :: m : n ::$ the velocity of the point F : the angular velocity of the quadrant.

Then, since the velocities are uniform,

$$z \ : \ AR \ :: \ m \ : \ n$$
$$AR \ : \ MF \ :: \ r \ : \ y$$
$$\therefore \ z \ : \ MF \ :: \ rm \ : \ ny.$$

Hence $MF = \dfrac{n}{mr} \times yz$, and the fluxion of the spherical

area $PMF = \dfrac{n}{mr} \times yz\dot{z} = \dfrac{n}{mr} \times - zrx\dot{x} = - \dfrac{n}{m} \times z\dot{x}$;

and the fluent or the area $PMF = - \dfrac{n}{m} \times (zx - fx\dot{z}) =$

$- \dfrac{n}{m} \times (zx - ry) = \dfrac{n}{m} \times (ry - zx)$, which is correct since the area $= 0$, when $y = 0$, and $z = 0$.

Cor. 1. Let the quadrant PMA make a complete revolution, in the same time that the point F describes a quadrant, which is the case considered by Pappus, and in which

$m : n :: 1 : 4$; then the spherical area $= \dfrac{n}{m} \times ry = \dfrac{n}{m} \times r^2 =$

$4r^2 = AB^2$, since the quantity zx in this case vanishes.

Cor. 2. In any other case, the area PAa : the square of the diameter $:: n : 4m ::$ arc Aa : the whole circumference.

Cor. 3. Hence a portion of the spherical surface included by the quadrant PMA, and the spiral line

admits of a perfect quadrature, when the ratio of the arc *Aa*, to the whole circumference can be assigned.

ON THE LOGARITHMIC SPIRAL.

(128.) DEF. Let *ABD* be a circle described with a given radius *SA*; then, as the arc *AB* of that circle
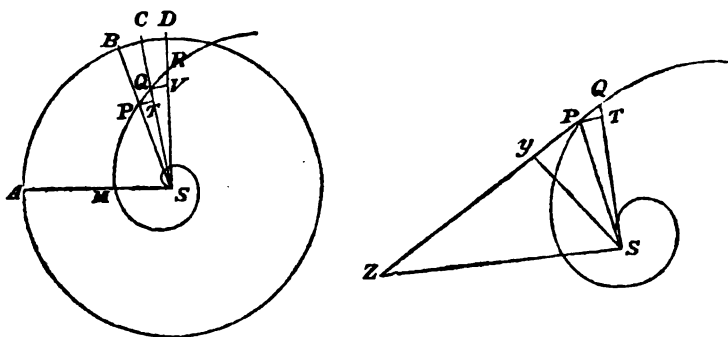
increases in arithmetic progression, if the distance *SP* increase in geometric, the point *P* will trace out the logarithmic spiral.

COR. 1. Let *SP*, *SQ*, *SR* be assumed indefinitely near each other, and equidistant, so that the arc *BC*= *CD*. Draw *PT*, *QV*, perpendicular to *SQ*, *SR*. Then, since $SP : SQ :: SQ : SR$, dividendo $SP : QT :: SQ : RV$, or $QT \propto SP$. That is, if $SP = y$; since QT ultimately becomes $\dot{y}$, $\dot{y} \propto y$.

COR. 2. The distance *SP* being supposed to revolve uniformly round *S*, the angle *PSQ* is constant, or $\dfrac{PT}{SP}$ is a constant quantity; hence, $PT \propto SP \propto QT$.

PROBLEMS.

(129.) PROB. 1. To draw a tangent to the logarithmic spiral at any point P.

The subtangent $SZ = \dfrac{PT \times y}{\dot{y}}$ (Art. 111.) Now PT :

QT ($\dot{y}$) in a given ratio, :: $a : b$; $\therefore SZ = \dfrac{ay}{b}$. Hence,

draw SZ perpendicular to SP, and equal to $\dfrac{ay}{b}$; join ZP; it is the tangent required.

COR. 1. $SP : SZ :: y : \dfrac{ay}{b} :: b : a$, a given ratio;

$\therefore$ the triangle SPZ is always similar to itself in the same spiral, and SPZ is a constant angle.

COR. 2. If Sy be drawn perpendicular to PZ, the ratios of $SP : Sy$, of $SP : Py$, and of $Sy : Py$, are given.

PROB. 2. To find the area contained between two rays, SP and SM.

The fluxion of a spiral area $= \dfrac{py\dot{y}}{2t}$; in this case, $p : t$ in a given ratio, :: $a : b$; $\therefore$ the fluxion of the area $= \dfrac{ay\dot{y}}{2b}$; and the fluent $= \dfrac{ay^2}{4b} + $ cor.; let $SP = SM = d$;

then the area $= \dfrac{a}{4b}(y^2 - d^2)$; or the area $SPM = \dfrac{a}{4b}$ $(SP^2 - SM^2.)$

COR. 1. The area contained between SP and $SM \propto SP^2 - SM^2$.

COR. 2. If $SM = 0$, or the whole spiral area be found between SP and the pole S, it $= \dfrac{a}{4b} \times SP^2 = \dfrac{Sy}{4Py} \times SP^2$

$= $ (since $Sy : Py :: SZ : SP$) $\dfrac{SZ}{4SP} \times SP^2 =$

$\dfrac{SZ \times SP}{4} = $ half the area of the triangle SPZ.

Prob. 3. To find its length.

The fluxion of the length $= \dfrac{y\dot{y}}{t}$. In this case, $y : t$ in

a constant ratio, $:: m : n;$ $\therefore \dfrac{y}{t} = \dfrac{m}{n}$, and $\dot{z} = \dfrac{m\dot{y}}{n};$

$\therefore z = \dfrac{my}{n} + $ cor. Let $SP = SM = d;$ then $PM = \dfrac{m}{n}$

$(SP - SM) = \dfrac{SP}{Py} (SP - SM)$.

Cor. 1. The arc $PM \propto SP - SM$.

Cor. 2. The whole arc to the pole $S = \dfrac{SP^2}{Py} = $ (since

$ZP : SP :: SP : Py$) the tangent ZP.

Prob. 4. To find the radius and chord of curvature.

By Art. 119, the radius of curvature $= \dfrac{x\dot{x}}{\dot{p}}$, where $SP = x;$

that is, in this case, where $SP = y$, it $= \dfrac{y\dot{y}}{\dot{p}}$. But $y : p$,

in a given ratio, $:: m : r;$ $\therefore \dfrac{\dot{y}}{\dot{p}} = \dfrac{m}{r}$, and $\dfrac{y\dot{y}}{\dot{p}} = \dfrac{my}{r};$

or the radius of curvature $= \dfrac{m \times SP}{r}$.

Again, by Art. 119, the chord of curvature $= \dfrac{2p\dot{x}}{\dot{p}};$

in this case $= \dfrac{2p\dot{y}}{\dot{p}}$. But, since $y : p :: m : r;$ $\therefore y :$

$p :: \dot{y} : \dot{p}$, and $\dfrac{\dot{y}}{\dot{p}} = \dfrac{y}{p};$ hence, the chord of curvature

$= 2p \times \dfrac{y}{p} = 2y = 2SP$.

PROB. 5. To determine the evolute CT.

By the nature of the evolute, CP the radius of curvature $= CT$; and $CP = \dfrac{m \times SP}{r}$ (by the last Prob.), or $CP : SP :: m : r$.

But $SP : PR$ or $Sy :: m : r$; $\therefore$ $CP : SP :: SP : PR$; hence (Euclid, B. VI.) the triangles CPS, SPR are similar; $\therefore$ PSC is a right angle, and the angle $SCP =$ the angle SPy; and $SC : CP$ in the given ratio of Py to SP. Therefore the evolute CT is a logarithmic spiral similar to SvP.
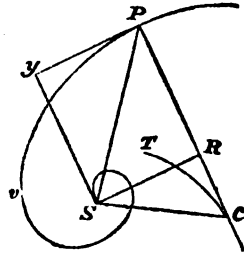
COR. A line SC drawn from S perpendicular to SP passes through C, the center of curvature.

Chap. XVI.

ON THE CONCHOID OF NICOMEDES.

(130.) Def. LET *CM* be a line of indefinite length, given in position; if about some point *B* taken without it, an indefinite line *BR* revolve, cutting *CM* in *v*, and *vR* be always taken of the same length, the point *R* will trace out the conchoid.
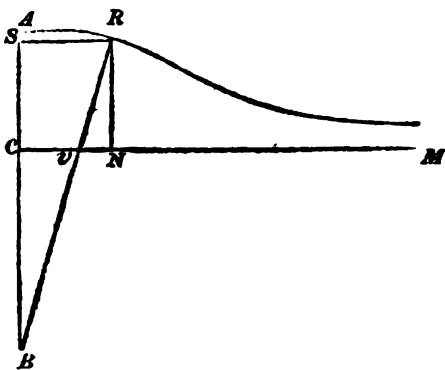
Problems.

(131.) Prob. 1.   To find the equation.

Draw *BCA* and *RN* perpendicular to *CM*, and *RS* perpendicular to *AB*. Take $BC=a$, $CA=vR=b$, $CN=x$, $NR=y$; then, by similar triangles, *BSR*, *vRN*,

$$BS : SR :: RN : Nv, \text{ or } a+y : x :: y : Nv = \frac{xy}{a+y};$$

and $vR^2=vN^2+NR^2$, or $b^2=y^2+\dfrac{x^2y^2}{(a+y)^2}$; $\therefore (a+y)^2 \times b^2=$

$(a+y)^2 \times y^2+x^2y^2$, or $(a+y)^2 . (b^2-y^2)=x^2y^2$, the equation.

Prob. 2.   To draw a tangent to the conchoid.

Since by the equat^n. $xy = (a+y) \sqrt{b^2-y^2}$; $\therefore x\dot{y}+y\dot{x}=$

$$\dot{y} \times \sqrt{b^2 - y^2} - \frac{y\dot{y}(a+y)}{\sqrt{b^2-y^2}} \; ; \; \therefore y\dot{x} = \dot{y} \times \sqrt{b^2 - y^2} - \frac{y\dot{y}(a+y)}{\sqrt{b^2-y^2}}$$

$$- x\dot{y} = \left(\text{since } x = \frac{(a+y)\sqrt{b^2-y^2}}{y} \right) \dot{y} \times \sqrt{b^2 - y^2}$$

$$- \frac{y\dot{y}(a+y)}{\sqrt{b^2-y^2}} - \frac{(a+y)\sqrt{b^2-y^2} \times \dot{y}}{y} = \text{by reduction}$$

$$- \frac{y^3 + ab^2}{y \times \sqrt{b^2-y^2}} \times \dot{y} \; ; \; \text{hence the subtangent, or } \frac{y\dot{x}}{\dot{y}} =$$

$$- \frac{y^3 + ab^2}{y \times \sqrt{b^2 - y^2}}, \text{ a negative quantity; therefore the tan-}$$

gent and the vertex are on contrary sides of the ordinate,

and the value of the subtangent, thus taken, is $\dfrac{ab^2 + y^3}{y \times \sqrt{b^2-y^2}}$.

PROB. 3. To find the area.

By the preceding Prob. $y\dot{x} = \dfrac{-y^3\dot{y}}{y \times \sqrt{b^2-y^2}} - \dfrac{ab^2\dot{y}}{y\sqrt{b^2-y^2}} =$

$\dfrac{-y^2\dot{y}}{\sqrt{b^2-y^2}} - \dfrac{ab^2\dot{y}}{y \times \sqrt{b^2-y^2}}$; and the area or $f \cdot y\dot{x} = \text{corr.} - \frac{1}{2}b$

$\times$ a circular arc of radius b, and sine $y + \dfrac{y \times \sqrt{b^2-y^2}}{2}$

(Fluent 15.) $- \dfrac{ab}{2} \times \text{hyp. log.} \dfrac{b - \sqrt{b^2-y^2}}{b + \sqrt{b^2-y^2}}$ (Art. 43).

Now the area = 0, when $y = b$; therefore the area
$ARNC =$ ($\frac{1}{2}b \times$ a circular arc of radius b, and sine b,
or) $\frac{1}{2}b \times$

$\left(\text{arc of a quad.} - \text{arc whose rad. is } b \text{ and sine } y + \dfrac{y \times \sqrt{b^2-y^2}}{2} \right)$

$- \dfrac{ab}{2} \times \text{hyp. log.} \dfrac{b - \sqrt{b^2-y^2}}{b + \sqrt{b^2-y^2}}$.

PROB. 4. To find the content of the solid generated
by the revolution of the conchoid about its axis CM.

Since $y\dot{x} = \dfrac{-y^3\dot{y}}{y \times \sqrt{b^2-y^2}} - \dfrac{ab^2\dot{y}}{y \times \sqrt{b^2-y^2}}$; $\therefore py^2\dot{x}$, or

the fluxion of the content $= -p \times : \dfrac{y^3\dot{y}}{\sqrt{b^2-y^2}} + \dfrac{ab^2\dot{y}}{\sqrt{b^2-y^2}}$;

and the content itself $=$ cor. $+ p \times \dfrac{2(b^2-y^2)^{\frac{3}{2}}}{3} + py^2\times$

$\sqrt{b^2-y^2}$ ($f.$ 17.) $-pab \times$ a circular arc of radius b and sine y (Art. 44.). Let $y = b$, the content $= 0$; $\therefore$ the correct fluent is pab. (the arc of a quadrant—arc of radius

b and sine y) $+ \dfrac{2p}{3}.(b^2-y^2)^{\frac{3}{2}} + py^2 \times \sqrt{b^2-y^2}$, the

content of the solid generated by the revolution of $ARNC$ about CN. Let $y = 0$; then we get the content of the whole solid, when its axis is infinite $= pab \times$ the arc of a

quadrant $+ \dfrac{2pb^3}{3} = pb^2\left(\dfrac{1}{2}pa + \dfrac{2b}{3}\right)$.

PROB. 5. To find the point of contrary flexure.

Since $y\dot{x} = -\dfrac{y^3+ab^2}{y \times \sqrt{b^2-y^2}} \times \dot{y}$; $\therefore \dot{x} = -\dfrac{y^3+ab^2}{y^2 \times \sqrt{b^2-y^2}} \times \dot{y}$.

Assume $\dot{y}$ constant or $= 1$; then, $\ddot{x} = \dfrac{2b^4a - b^2y^3 - 3b^2ay^2}{(b^2y^2-y^4)\sqrt{b^2-y^2}} = 0$;

$\therefore 2b^4a - 3b^2ay^2 - b^2y^3 = 0$, or $y^3 + 3ay^2 - 2ab^2 = 0$, an equation from which the value of y, and therefore of x, may be determined.

If $a = b$, then the equation becomes $y^3 + 3by^2 - 2b^3 = 0$; or dividing by $y + b$, it becomes $y^2 + 2by - 2b^2 = 0$; hence $y (NR = CS) = -b + b\sqrt{3} = b(-1+\sqrt{3}) = .73205\ b$. And in the same case $BR = \frac{1}{2}b\ (3 + \sqrt{3}) = 2.366\ b$.

ON THE CISSOID OF DIOCLES.

(132.) Let AB be the diameter of a semi-circle ACB, D and M two points in it equally distant from B and A; from D and M draw DC, MP, perpendicular to AB; and join AC, cutting MP in P. The point P traces out the Cissoid.
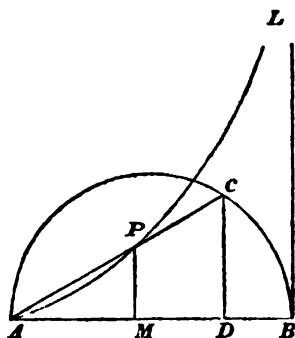
Cor. BL drawn perpendicular to AB is an asymptote to the curve AP.

PROBLEMS.

(133.) Prob. 1. To find the equation to the cissoid of Diocles.

By the properties of the circle, $AD \times DB = DC^2$; but $AD \times DB = AM \times MB$; $\therefore AM \times MB = DC^2$. Also

$$AM : MP :: (AD)\ MB : DC; \therefore DC = \frac{MP \times MB}{AM};$$

$$\therefore DC^2 = \frac{MP^2 \times MB^2}{AM^2}; \text{ consequently } \frac{MP^2 \times MB^2}{AM^2} =$$

$AM \times MB$; $\therefore MP^2 \times MB = AM^3$; that is, if $AB = a$, $AM = x$, $MP = y$, $y^2 (a - x) = x^3$.

Prob. 2. To draw a tangent to the cissoid.

The equat^n. is $y^2 = \dfrac{x^3}{a-x}$; $\therefore 2y\dot{y} = \dfrac{3x^2\dot{x}(a-x) + x^3\dot{x}}{(a-x)^2} =$

$\dfrac{(3ax^2 - 3x^3 + x^3)\dot{x}}{(a-x)^2}$; $\therefore \dfrac{\dot{x}}{\dot{y}} = \dfrac{2y(a-x)^2}{3ax^2 - 2x^3}$; $\therefore \dfrac{y\dot{x}}{\dot{y}}$ the sub-

$$\text{tangent} = \frac{2y^2\,(a-x)^2}{3\,ax^2 - 2x^3} = \frac{2\,x^3\,(a-x)}{3\,ax^3 - 2x^3} = \frac{2\,x\,(a-x)}{3\,a - 2x}\ .$$

PROB. 3. To find the area.

Since $y = \dfrac{x^{\frac{3}{2}}}{\sqrt{a-x}}$; $\therefore y\dot{x} = \dfrac{x^{\frac{3}{2}}\dot{x}}{\sqrt{a-x}} = \dfrac{x^2\dot{x}}{\sqrt{ax-x^2}}$, whose

fluent $(f.\,23.) = \dfrac{3a}{4} \times$ a circular arc of radius $= \frac{1}{2}\,a$, and

versed sine $x - x \times \dfrac{\sqrt{ax - x^2}}{2} - \dfrac{3a}{4} \times \sqrt{ax - x^2}$; which

vanishes when $x = 0$; and when $x = a$, it $= \dfrac{3a}{4} \times$ the arc

of a semi-circle; $\therefore$ for both sides of the axis it $= \dfrac{3a}{2} \times$ the

arc of a semi-circle $=$ three generating circles: so that
though the dimensions are infinite, the area is finite.

PROB. 4. To find the content of the solid generated
by the rotation of the cissoid about its axis AB.

The $f.\,py^2\dot{x} = f.\,\dfrac{p\,x^3\dot{x}}{a-x} = -\ \frac{1}{3}\,px^3 - \frac{1}{2}\,pax^2 - pa^2x + pa^3 \times$

hyp. log. $\dfrac{a}{a-x}$. (Fluent 3.)

ON THE LOGARITHMIC CURVE.

DEFINITION.

(134.) If on the indefinite
line AL, the parts AB, BC,
CD, DE, &c. be taken equal
to each other, and ordinates
AF, BG, CH, &c. be drawn
perpendicular to AL, and in
geometrical progression, the curve FHK, which passes
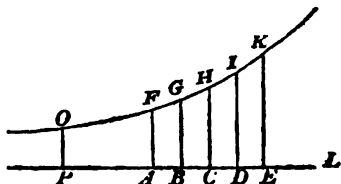
through all their extremities, is called the Logarithmic Curve.

COR. The axis LA produced, is an asymptote to the curve; for since the terms of a decreasing geometric series never become accurately equal to nothing, a distant ordinate, as OP, never is accurately equal to nothing, though it decreases *sine limite*.

PROBLEMS.

(135.) PROB. 1. To find the equation to this curve.

By the nature of logarithms, any abscissa AD is the logarithm of the ordinate DI, in a system which depends upon the magnitude of AF, and BG; the part AB being given. Let $AB=1$, the logarithm of BG; then if $BG=a$, $AD=x$, $DI=y$,

$1 : x :: $ log. of a : log. of y; but $1 = $ log. of a; $\therefore x = $ log. of y; that is, $x \times 1$, or $x \times$ log. of a, $= $ log. of y; $\therefore y = a^x$ the equation.

PROB. 2. To draw a tangent to the logarithmic curve.

Let $y = a^x$; $\therefore$ log. $y = x \times$ log. a; and $\dfrac{\dot{y}}{y} = \dot{x} \times$ log. a;

$\therefore \dfrac{y\dot{x}}{\dot{y}} = \dfrac{1}{\text{log. } a} = $ a constant quantity.

COR. 1. If $\dfrac{1}{\text{log. } a} = m$, $\dfrac{y\dot{x}}{\dot{y}} = m$; and $y\dot{x} = m\dot{y}$, which may be taken as another equation to the curve.

COR. 2. In any system, where x is the logarithm of y to a modulus m, $\dot{x} = m \times \dfrac{\dot{y}}{y}$ (Art. 41.); hence, in the logarithmic curve, the sub-tangent is the modulus of the system.

PROPOSITION.

(136.) If two ordinates in one logarithmic curve be in the

same ratio with two ordinates in another, the abscissas are as the sub-tangents.

Let AE, ae be the abscissas, ET and et the sub-tangents, KE, ID in one curve, and ke, id in the other,
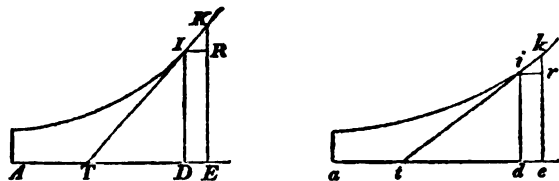
ordinates indefinitely near to each other, and in a given ratio;

Then $KE : ID :: ke : id$, by supposition;

$\therefore KE : KR :: ke : kr$, or $y : \dot{y} :: Y : \dot{Y}$;

$\therefore \dfrac{\dot{y}}{y} = \dfrac{\dot{Y}}{Y}$; but $\dot{x} = m \times \dfrac{\dot{y}}{y}$; $\therefore \dot{x}$ in one case : $\dot{X}$ in the

other $:: m \times \dfrac{\dot{y}}{y} : M \times \dfrac{\dot{Y}}{Y} :: m : M$, and $x : X :: m : M$,

or $AE : ae :: TE : te$.

PROB. 3. To find the area included between any two given ordinates, as BG, EK.

Since $y\dot{x} = m\dot{y}$; $\therefore f. y\dot{x} = my +$
corr. Let the area $= 0$, when $y = BG = a$; then the area included between BG and $EK = my - ma = m \times (EK - BG.)$
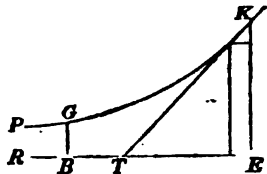
COR. 1. If BG go off *ad infinitum*, and become evanescent, the whole area included between the asymptote ER and the curve $= m \times EK =$ sub-tangent $\times KE =$ twice the triangle EKT.

COR. 2. In the same logarithmic curve, the area included between any two ordinates : the area included between any other two :: the difference of the two first : the difference of the two last.

COR. 3. The whole area included from KE between the asymptote and the curve varies as KE.

PROB. 4. To find the content of the solid generated by the revolution of the area $BGKE$ about its axis BE.

Since $y\dot{x} = m\dot{y}$; $\therefore f. py^2\dot{x} = f. mpy\dot{y} = mp\,\dfrac{y^2}{2} +$ corr. But the content $= 0$, if $y = BG = a$; $\therefore$ the content corrected $= \dfrac{mp}{2}(y^2 - a^2)$; and the content included between BG and $EK = \dfrac{mp}{2}(KE^2 - BG^2)$.

COR. 1. Suppose GB to go off *ad infinitum*; then the content of the whole solid generated by the revolution of the infinitely extended area $KPRE$ about its axis, $= \dfrac{mp}{2} \times KE^2 = \dfrac{m}{2} \times$ the area of a circle, whose radius is $KE = \frac{1}{2}$ the content of a cylinder whose altitude is the sub-tangent, and radius of the base equal to the ordinate of the curve.

COR. 2. The solid generated by the revolution of $BGKE = \frac{1}{4}$ the cylinder, whose altitude is the sub-tangent, and base the difference of the circles, whose radii are EK and BG.

COR. 3. The content of the infinitely extended solid varies as KE^2, where the sub-tangent is the same.

PROB. 5. To find the length of the logarithmic curve included between any two ordinates GB, KE.

Here $\dot{z} : \dot{y} :: KT : KE :: \sqrt{m^2 + y^2} : y$; therefore $\dot{z} = \dfrac{\dot{y} \times \sqrt{m^2 + y^2}}{y}$; $\therefore$ by $f.$ 60, z or $KG = \sqrt{m^2 + y^2} + \dfrac{m}{2} \times$ hyp. log. $\dfrac{\sqrt{m^2 + y^2} - m}{\sqrt{m^2 + y^2} + m} - \sqrt{m^2 + a^2} - \dfrac{m}{2} \times$ hyp. log. $\dfrac{\sqrt{m^2 + a^2} - m}{\sqrt{m^2 + a^2} + m}$, corrected for the value of $y = a$.

PROB. 6. To find the surface of the solid generated by the revolution of any part KG of the logarithmic curve about its axis.

By the last Example, $\dot{z} = \dfrac{\dot{y} \times \sqrt{m^2 + y^2}}{y}$; $\therefore 2py\dot{z}$, the fluxion of the surface, $= 2p \times \dot{y} \times \sqrt{m^2 + y^2}$, of which the fluent, (by $fl.$ 58.) when properly corrected, $= p \times \sqrt{y^4 + m^2 y^2} - p \times \sqrt{a^4 + m^2 a^2} + pm^2 \times$ hyp. log. $\dfrac{y + \sqrt{m^2 + y^2}}{a + \sqrt{m^2 + a^2}} =$ the surface required; the values of a and m being the same as before.

PROB. 7. To find the evolute of the logarithmic curve.

Let AKD be the curve; KT a tangent at K; ET the sub-tangent; EH the sub-normal. Then if C be assumed as the center of the circle of curvature at K, and CN be drawn perpendicular to EK produced, KN corresponds
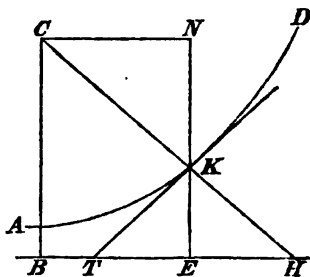
with the line $\frac{1}{2}ME$, in Art. 100; therefore $KN = \dfrac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}}$.

But $\dot{y} = \dfrac{y\dot{x}}{m}$; $\therefore$ if $\dot{x}$ be constant, $\ddot{y} = \dfrac{\dot{y}\dot{x}}{m} = \dfrac{y\dot{x}^2}{m^2}$; and by substitution, $KN = \dfrac{m^2 + y^2}{-y}$. But $EH = \dfrac{y^2}{m}$; $\therefore$ by similar triangles, KEH, CKN, $CN = \dfrac{m^2 + y^2}{-m} = -m$ $-\dfrac{y^2}{m} = -TE - EH$. Hence, take $EB = TH$, and draw BC perpendicular to BE; it will meet the normal in C, the required point of the evolute.

ON THE CATENARY.

(137.) To find the curve, into which a flexible chain of uniform density and thickness would form itself, if suffered to hang freely from two points, A and B. ·

Let ADB be the curve, D the lowest point; draw the axis DC perpendicular to the horizon; let Ht, Fo, be two ordinates, ts a tangent at t, and tr parallel to CD. Now the chain having assumed this form, it is immaterial whether the part Dt be considered as flexible or rigid. On the last supposition, it is kept at rest by three forces; by BD in the direction DE or sF, by At in the direction ts, and by its gravity in the direction rt; therefore, by Mechanics, these forces are as the sides of the triangle str. Let the tension of $BD = a$, $Dt = z$, $DH = x$, $Ht = y$; then $a : z :: \dot{y} : \dot{x}$; $\therefore a\dot{x} = z\dot{y}$,
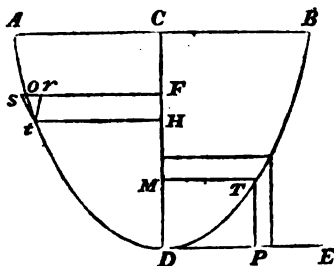
$$\dot{y}^2 = \frac{a^2\dot{x}^2}{z^2}; \quad \therefore \dot{z}^2, \text{ or } \dot{x}^2 + \dot{y}^2, = \dot{x}^2 + \frac{a^2\dot{x}^2}{z^2}; \text{ hence,}$$

$$z\dot{z} = \sqrt{z^2 + a^2} \times \dot{x}; \quad \therefore \dot{x} = \frac{z\dot{z}}{\sqrt{z^2 + a^2}}; \text{ and } x = \sqrt{z^2 + a^2}$$

+cor. Let $x = 0$, $z = 0$; $\therefore$ cor. $= -a$; $\therefore x = \sqrt{z^2 + a^2} - a$; $\therefore a + x = \sqrt{z^2 + a^2}$, and $x^2 + 2ax = z^2$, the equation to the curve.

Cor. Since $\dot{y} = \frac{a\dot{x}}{z} = \frac{a\dot{x}}{\sqrt{2ax + x^2}}$; $y = a \times$ hyp. log. $(x + a + \sqrt{2ax + x^2}.)$

PROBLEMS.

PROB. 1. To draw a tangent to the catenary.

By the nature of the curve, $a\dot{x} = z\dot{y}$; $\therefore \dfrac{\dot{x}}{\dot{y}} = \dfrac{z}{a}$; $\therefore \dfrac{y\dot{x}}{\dot{y}} =$

$$\dfrac{y \times z}{a} = \dfrac{y \times \sqrt{2ax \times x^2}}{a} = \text{ the sub-tangent.}$$

PROB. 2. To find its area. (See the preceding FIGURE.)
By the equation, $z\dot{y} = a\dot{x}$;

$$\therefore z^2\dot{y}^2 = a^2\dot{x}^2 = a^2(\dot{z}^2 - \dot{y}^2) = a^2\dot{z}^2 - a^2\dot{y}^2 ;$$
$$\therefore (z^2 + a^2)\dot{y}^2 = a^2\dot{z}^2,$$
$$\text{or } (a+x)^2 \times \dot{y}^2 = a^2\dot{z}^2 ;$$
$$\therefore (a+x)\dot{y} = a\dot{z},$$
$$\text{or } a\dot{y} + x\dot{y} = a\dot{z} ;$$

$\therefore x\dot{y}$, the fluxion of the external area $DTP, = a\dot{z} - a\dot{y}$;
and the area $BDT = az - ay$, and cor. $= 0$; $\therefore DMT =$
$xy - az + ay = (a+x)\,y - a\sqrt{2ax + x^2}.$

PROB. 3. To find the content of the solid generated
by the revolution of the catenary about its axis.

The fluent of $py^2\dot{x} = py^2x - f.\,2pxy\dot{y}$; and the fluent
of $2py \times x\dot{y} =$ the fluent of $2pay\dot{z}$ – the fluent of $2pay\dot{y}$;
since, by the last Article, $x\dot{y} = a\dot{z} - a\dot{y}$;

$$\therefore f.\,2pxy\dot{y} = f.\,2pay\dot{z} - pay^2 ;$$
$$\text{and } f.\,2pay\dot{z} = 2payz - f.\,2paz\dot{y} ;$$
$$= 2payz - 2pa^2x, \text{ by writing}$$

for $z\dot{y}$ its equal $a\dot{x}$; $\therefore$ the content $= py^2x + pay^2 - 2payz$
$+ 2pa^2x$, which needs no correction.

PROB. 4. To find the surface of the solid generated by
the revolution of the catenary.

The fluent of $2py\dot{z} = 2pyz - f.\,2pz\dot{y}$. But $z\dot{y} = a\dot{x}$;
$\therefore$ the fluent $= 2pyz - 2pax$, which needs no correction.

PROB. 5. To find the radius of curvature of the
catenary.

By Art. 100, the radius of curvature $= \dfrac{\dot{z}^3}{-\ddot{y}}$, when $\dot{x}$

is constant, and $= 1$. Now $z = \sqrt{2ax + x^2}$; $\therefore \dot{z} =$

$\dfrac{a\dot{x} + x\dot{x}}{\sqrt{2ax + x^2}} = \dfrac{a + x}{\sqrt{2ax + x^2}}$; $\therefore \dot{z}^3 = \dfrac{(a+x)^3}{(2ax + x^2)^{\frac{3}{2}}}$. Also

$\dot{y} = \dfrac{a\dot{x}}{\sqrt{2ax + x^2}} = \dfrac{a}{\sqrt{2ax + x^2}}$; $\therefore -\ddot{y} = a \times \dfrac{a + x}{(2ax + x^2)^{\frac{3}{2}}}$;

$\therefore$ the radius of curvature $= \dfrac{(a+x)^3}{a(a+x)} = \dfrac{(a+x)^2}{a}$.

Cor. At the vertex, where $x = 0$, the radius of curvature $= a$.

CHAP. XVII.

ON THE ATTRACTIONS OF BODIES.

PROBLEMS.

(138.) PROB. 1. To determine the attraction of a corpuscle P towards a right line AB, in a direction perpendicular to that line; the attracting force of each particle being supposed to vary inversely as the square of the distance.

Let PA perpendicular to $AB = a$; AC, a variable part of AB, $= x$; then $PC = \sqrt{a^2 + x^2}$. Now, since the force of attraction varies as $\dfrac{1}{\overline{\text{dist.}}|^2}$, the force which draws P toward C may be represented by $\dfrac{1}{PC^2}$. But the force of attraction toward C: that toward A by resolution of force :: $PC : PA :: \sqrt{a^2 + x^2} : a$; therefore the force of attraction toward A for one particle $C = \dfrac{a}{(a^2 + x^2)^{\frac{3}{2}}}$. And if the number of attracting particles at the distance $x = \dot{x}$, the attraction from these particles will be $\dot{x}$ times as great; hence the fluxion of the attraction to $AC = \dfrac{a \dot{x}}{(a^2 + x^2)^{\frac{3}{2}}}$; and its fluent, or $\dfrac{x}{a \times \sqrt{a^2 + x^2}}$, is the attraction to AC; and the attraction of P to the whole line $AB = \dfrac{AB}{PA \times PB}$.
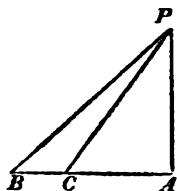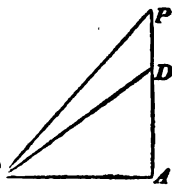
PROB. 2. To find the attraction of a line PA to AB; PA being perpendicular to AB, and the force of each particle varying as $\dfrac{1}{\overline{\text{dist.}}|^2}$.

Let $AB=a$, AD a variable part of $AP=x$; join BD. Then, by the last Case, the attraction of a corpuscle D to $AB = \dfrac{a}{x \times \sqrt{a^2 + x^2}}$.

Therefore the fluxion of this attraction

$= \dfrac{a\dot{x}}{x \times \sqrt{a^2 + x^2}}$; and the fluent, or the

attraction itself, $= \frac{1}{2}$ hyp. log. $\dfrac{\sqrt{a^2 + x^2} - a}{\sqrt{a^2 + x^2} + a}$ + corr. (Art.

43.)$= \frac{1}{2}$ hyp. log. $\dfrac{\sqrt{a^2 + x^2} - a}{\sqrt{a^2 + x^2} + a} - \frac{1}{2}$ hyp. log. $\dfrac{0}{2a}$, when

$x = PA = \frac{1}{2}$ hyp. log. $\dfrac{BP - BA}{BP + BA} - \frac{1}{2}$ hyp. log. $\dfrac{0}{2AB}$, an infinite quantity.

PROB. 3. Let C be the center of a circle ABE, and a corpuscle P be situate in the line PC perpendicular to its plane; it is required to determine the attraction of P to the circle, the force of each particle varying as $\dfrac{1}{D^2}$.

Let $PC=a$, $PD=x$; then $CD=\sqrt{x^2 - a^2}$, and the force of attraction of P to D being represented by $\dfrac{1}{PD^2}$, that in the direction $PC = \dfrac{PC}{PD^3}$ (Art. 138. Prob. 1.) Now if $p=3.14159$ &c. the area of a circle whose radius is 1, the area of the circle of
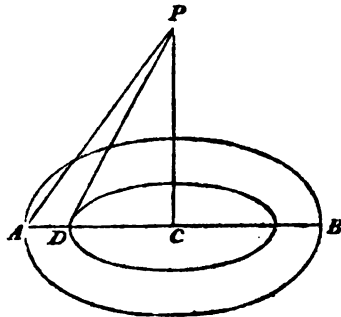
radius $CD = p\ (x^2 - a^2)$; and its fluxion $= 2px\dot{x}$; hence the fluxion of the attraction of P toward the circle $= \dfrac{a}{x^3} \times 2px\dot{x} = 2pax^{-2}\dot{x}$; whose fluent $= 2pa \times -\dfrac{1}{x}$

$+$ cor. $= 2pa\left(\dfrac{1}{a} - \dfrac{1}{x}\right) = 2p\left(1 - \dfrac{a}{x}\right) =$ for the whole

circle $2p\left(1 - \dfrac{PC}{PA}\right)$.

Cor. The force of attraction of P to the whole circle varies as $1 - \dfrac{PC}{PA}$.

Prob. 4. To determine the same, when the force of attraction varies as $\overline{D}|^n$.

Here, as before, $2px\dot{x}$ is the fluxion of the variable circle, and $\dfrac{a}{x} \times x^n$ or ax^{n-1} represents the force of a particle D, in the direction PC; hence, the fluxion of the attraction to the circle $= 2pax^n\dot{x}$; and the fluent $= 2pa \times \dfrac{x^{n+1}}{n+1} +$

cor. $= 2pa \times \dfrac{x^{n+1} - a^{n+1}}{n+1}$, which varies as $\dfrac{ax^{n+1} - a^{n+2}}{n+1}$, or

varies for the whole circle as $\dfrac{PC \times PA^{n+1} - PC^{n+2}}{n+1}$.

Cor. 1. If the diameter of the circle be increased in infinitum, and n be a negative number greater than 1, the attraction varies as $-\dfrac{PC}{PA^{n-1}} + \dfrac{1}{PC^{n-2}}$, or as $\dfrac{1}{PC^{n-2}}$, because the first term vanishes.

Cor. 2. If the radius of the circle be infinite, and the force vary as $\dfrac{1}{D^2}$, the attraction varies as $\dfrac{1}{PC^0}$; or is constant, whatever be the length of PC.

Prob. 5. Let P represent a corpuscle placed in the

vertex of a cone PAB; to determine the attraction to the cone by a force varying as $\dfrac{1}{D^2}$.

The attraction to the circle DEF varies as $1 - \dfrac{PE}{PF}$; and

since $\dfrac{PE}{PF}$ in the cone is a constant quantity, the attraction to every section parallel to the base is the same; hence, the whole attraction to the cone varies as $\left(1 - \dfrac{PE}{PF}\right)$. the number of sections, or as $\left(1 - \dfrac{PC}{PB}\right).PC.$
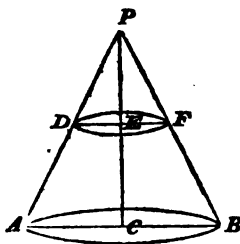
Cor.   For similar cones, the attraction is as the height.

Prob. 6.   The law of the force remaining the same, to find the attraction of a cylinder $ABCD$ upon a corpuscle P situated in its axis produced.

Let $PM = x$, $MF = a$, and $PF = \sqrt{a^2 + x^2}$. The attraction to the circle EMF is as $1 - \dfrac{PM}{PF}$, or as $1 - \dfrac{x}{\sqrt{a^2 + x^2}}$; $\therefore$ the fluxion of the attraction is proportional to $\dot{x} - \dfrac{x\dot{x}}{\sqrt{a^2 + x^2}}$; and the $\mathit{fl.} = x - \sqrt{a^2 + x^2} +$ cor. $= PM - PF +$ cor. $=$ $PM - PF - PH + PD$, or for the whole cylinder $=$ $PL - PH - PC + PD = HL - PC + PD.$
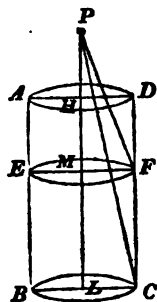
Cor.   If the diameter be infinite, $PC = PD$; and the attraction is proportional to HL.

Prob. 7.   To find the attraction of a corpuscle P to a sphere, the attractive force of each particle varying as $\dfrac{1}{D^2}$.

K

Draw PFB passing through F the center of the sphere, and let CED be a section of the sphere perpendicular to AB. Take $AF=a$, $PF=b$, $PA=c=b-a$, $PE=y$, $PC=c+x$; then, $AE=y-c$, and $EB=2a-y+c$. Now $AE \times EB$ $=EC^2=PC^2-PE^2$; that is, $(y-c) \cdot (2a-y+c) = (c+x)^2 - y^2$; from which equation $y=\dfrac{2ac+2c^2+2cx+x^2}{2a+2c}$
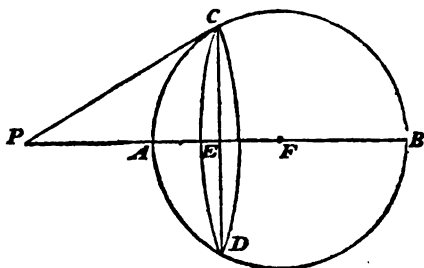
(since $b = a + c$)$=\dfrac{2bc+2cx+x^2}{2b}$. Hence the attraction to the circle is as $1 - \dfrac{2bc+2cx+x^2}{2b(c+x)}$, or as $\dfrac{2ax-x^2}{b(c+x)}$; and the fluxion of the attraction to the sphere is proportional to $\dfrac{2ax-x^2}{b(c+x)} \times \dot{y}$, that is, to $\dfrac{2ax-x^2}{b(c+x)} \times \dfrac{(c+x) \cdot \dot{x}}{b}$, or to $\dfrac{2ax\dot{x}-x^2\dot{x}}{b^2}$; whose fluent $=\dfrac{ax^2 - \frac{1}{3}x^3}{b^2}$, which needs no correction. Let $x = 2a$; then the attraction to the whole sphere is as $\dfrac{4a^3}{3b^2}$, or varies as $\dfrac{a^3}{b^2}$.

COR. 1. If the corpuscle be at the surface of the sphere, or if b be any multiple of a, the attraction varies as a.

COR. 2. Since the contents of spheres vary as a^3, the attraction varies as the content divided by the square of the distance from the center; and is the same as if all the matter of the sphere were collected into the center.

PROB. 8. To find the attraction of a particle P towards

a plane $LloO$ infinitely extended, the force of each particle varying inversely as the $\overline{\text{dist.}}]^{n}$.

CASE 1.

Let MHM represent a section of the infinite plane perpendicular to PHK; the attraction to the plane MHN, considered as circular, is as $\dfrac{1}{PH^{n-2}}$, P or if $PH=x$, as $\dfrac{1}{x^{n-2}}$; $\therefore$ the fluxion of the attraction is as $\dfrac{\dot{x}}{x^{n-2}}$; and the fluent $=$

cor. $-\dfrac{1}{(n-3)x^{n-3}} = \dfrac{1}{(n-3).PG^{n-3}} - \dfrac{1}{(n-3).PH^{n-3}}$;

when PH becomes infinite, it $= \dfrac{1}{(n-3).PG^{n-3}}$; or the whole attraction varies as $\dfrac{1}{PG^{n-3}}$.

CASE 2.

If P be placed within the plane, and PH be taken $= PG$, the particle P will be equally attracted each way towards the planes bounded by LGl, MHm, or they will destroy the effect of each other; hence, P may be considered as attracted only by the infinite plane $MHmoKO$; and the attraction to this plane by the last Case, is as $\dfrac{1}{PH^{n-3}}$,

or as $\dfrac{1}{PG^{n-3}}$.

Cor. 1.   If $n=3$, the fluent fails; for the fluxion of the attraction is as $\dfrac{\dot{x}}{x}$.

Cor. 2.   If the plane be finite, but PG extremely small, if compared with PH, the force of attraction is still proportional to $\dfrac{1}{PG^{n-3}}$.

Def.   An oblong spheroid is generated by the revolution of an ellipse round its major axis.

Prob. 9.   A particle of matter is placed in the pole A of an oblong spheroid, whose major axis : minor :: 1 : $1-n$, where n is very small; it is required to compare the attraction of this particle to the center of the spheroid, with its attraction to the center of the sphere described round the axis major, the force of each particle varying as $\dfrac{1}{D^2}$.

Let $AE=x$; then, $DE=(1-n).\sqrt{2x-x^2}$ by the property of the ellipse ; $\therefore AD^2=$
$x^2 + (1-n)^2.(2x-x^2)=x^2 +$
$(1-2n+n^2).(2x-x^2) = x^2 +$
$(1-2n).(2x-x^2)$, n^2 being omitted as indefinitely small,
$=x^2 + 2x - x^2 - 4nx + 2nx^2 =$
$2x - 4nx + 2nx^2$; $\therefore AD=$
$(2x - 4nx + 2nx^2)^{\frac{1}{2}}$.   Hence the attractive force to the circle, whose diameter is $2DE$, $=1 - \dfrac{x}{(2x-4nx+2nx^2)^{\frac{1}{2}}}$;
and the fluxion of the attraction of the spheroid is proportional to $\dot{x} - \dfrac{x\dot{x}}{(2x-4nx+2nx^2)^{\frac{1}{2}}}$; therefore the attraction itself is as the fluent of this quantity, or proportional to $x - \dfrac{2^{\frac{1}{2}}x^{\frac{3}{2}}}{3} - \dfrac{2^{\frac{1}{2}}nx^{\frac{3}{2}}}{3} + \dfrac{nx^{\frac{5}{2}}}{5\times2^{\frac{1}{2}}}$ (fl. 91.)   which needs
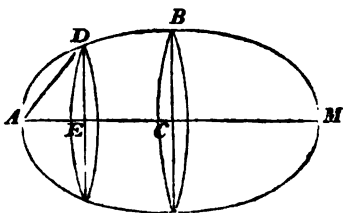
no correction. Let $x = 2$; then the attraction to the whole spheroid is proportional to $2 - \dfrac{4}{3} - \dfrac{4n}{3} + \dfrac{4n}{5}$, which varies as $\dfrac{2}{3} - \dfrac{8n}{15}$, or as $1 - \dfrac{4n}{5}$. If $n = 0$, the spheroid becomes a sphere; hence, the attraction of this spheroid on a particle at A : the attraction of the sphere described round AM on the same particle $:: 1 - \dfrac{4n}{5} : 1$.

Def. An oblate or compressed spheroid is generated by the revolution of an ellipse round the minor axis.

Prob. 10. A particle is placed in the pole B of an oblate spheroid, whose minor axis : major $:: 1 : 1+n$, where n is very small; it is required to compare the attraction of this particle toward the center of the spheroid, with its attraction toward the center of the sphere described round its minor axis, the force of each particle varying as $\dfrac{1}{D^2}$.

The ratio of the major to the minor axis is, in reality, the same in this case as in the last; for $1 : 1 - n :: 1 + n : 1 - n^2$ or 1, since n^2 is extremely small.
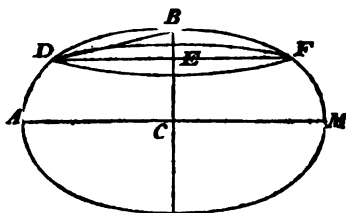
Let $BE = x$; then, $DE^2 = (1+n)^2 . (2x - x^2)$ by the property of the ellipse; $\therefore BD^2 = x^2 + (1+n)^2 . (2x - x^2) = x^2 + (1 + 2n + n^2) . (2x - x^2) = x^2 + (1 + 2n) . (2x - x^2) = x^2 + 2x + 4nx - 2nx^2 - x^2 = 2x + 4nx - 2nx^2$; $\therefore BD = (2x + 4nx - 2nx^2)^{\frac{1}{2}}$; and the attraction to the circle DEF is as $1 - \dfrac{x}{(2x + 4nx - 2nx^2)^{\frac{1}{2}}}$; $\therefore$ the fluxion of

the attraction to the spheroid $\propto \dot{x} - \dfrac{x\dot{x}}{(2x + 4nx - 2nx^2)^{\frac{1}{3}}}$;

and the fluent ($fl.$ 92.) $= x - \dfrac{2^{\frac{1}{2}}x^{\frac{2}{3}}}{3} - \dfrac{nx^{\frac{5}{3}}}{5 \times 2^{\frac{1}{3}}} + \dfrac{2^{\frac{1}{2}}nx^{\frac{5}{3}}}{3}$, which

needs no correction. Let $x = 2$; then the attraction to,

the whole spheroid is as $2 - \dfrac{4}{3} - \dfrac{4n}{5} + \dfrac{4n}{3}$, or as $\dfrac{2}{3} + \dfrac{8n}{15}$,

which varies as $1 + \dfrac{4n}{5}$. If $n = 0$, the spheroid becomes

a sphere round the axis minor; hence, the attraction of
the oblate spheroid on a particle at B : attraction of
the sphere described round the axis minor on the same

particle :: $1 + \dfrac{4n}{5}$: 1.

Cor. 1. Since these spheroids, by supposition, approach
nearly to spheres, they may, without sensible error, be
assumed for spheres containing the same quantity of
matter; and their attractions, *cœteris paribus*, will be
proportional to their quantities of matter. But (Art. 51.
Ex. 4. Cor. 2.) the oblong spheroid : the oblate :: the
oblate : the circumscribed sphere; hence, the attraction
of the oblong spheroid on a particle at A : the attraction
of the oblate on the same particle :: the attraction of the
oblate : the attraction of the circumscribed sphere upon
it. Therefore, the attraction of the oblate spheroid on a
particle at A : the attraction of the circumscribed sphere
on the same particle :: $\sqrt{}$ the attraction of the oblong
spheroid : $\sqrt{}$ the attraction of the circumscribed sphere

:: $\sqrt{1 - \dfrac{4n}{5}}$: 1 :: $1 - \dfrac{2n}{5}$: 1.

Cor. 2. By help of the preceding Propositions, since

the Earth is an oblate spheroid, we can determine the ratio of its polar and equatorial diameters.

Let ABL represent the Earth; AC and BC the equatorial and polar semi-diameters. Then by the preceding Cases, the attraction of the compressed spheroid on a particle at B : the attraction of the inscribed sphere upon
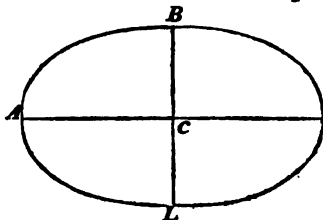
the same particle :: $1 + \dfrac{4n}{5}$: 1.

The attraction of the inscribed sphere on a particle at B : the attraction of the circumscribed sphere on the same at the surface, or at A, :: 1 : $1+n$. (Chap. xvii. Prob. 7. Cor. 1.)

The attraction of the circumscribed sphere on a particle at A : the attraction of the compressed spheroid on

an equal particle at A :: 1 : $1 - \dfrac{2n}{5}$.

Therefore the attraction of the compressed spheroid on a particle at B : the attraction of the same spheroid

on an equal particle at A :: $1 + \dfrac{4n}{5}$: $1 + \dfrac{3n}{5}$.

Now these attractions must be proportional to the weights of equal quantities of matter at those points; and if the particles are not equal, their weights are as the weights of equal particles, and their magnitudes jointly. Hence, if the axes are in the ratio of $1 - n$: 1, and particles be taken proportional to the axes in each, we have, the weight of $1 - n$ particles on the spheroid at B : the

weight of a particle at $A :: \left(1 + \dfrac{4n}{5}\right).(1-n) : 1 + \dfrac{3n}{5}$

$$:: 1 - \frac{n}{5} : 1 + \frac{3n}{5},$$

the powers of n being omitted. And dividendo,

The weight of a particle at A — the weight of $1 - n$ particles at B : the weight of a particle at $A :: \dfrac{4n}{5}$:

$1 + \dfrac{3n}{5}$.

Now if the figure of the Earth be caused by rotation round its axis BL, since $(1-n)$ times a particle at B must be in equilibrio with a particle at A, the excess of weight of one above the other must be counterbalanced by the centrifugal force, arising from rotation, and is consequently equivalent to it. Therefore, the difference of weight of $1-n$ particles at B, and a particle at A : the weight of a particle at A :: the centrifugal force at the equator : the force of gravity there ; that is, according to NEWTON (Lib. 3.) :: 1 : 289.

Hence, $\dfrac{4n}{5} : 1 + \dfrac{3n}{5} :: 1 : 289,$

or $\dfrac{4n}{5} \times 289 = \dfrac{5 + 3n}{5}$. From this equation,

$n = \dfrac{1}{230}$ nearly. Consequently, the polar diameter is to

the equatorial :: $1 - n : 1$, is as $1 - \dfrac{1}{230} : 1,$

$$\text{or} :: 229 : 230.$$

SCHOLIUM.

We have here supposed the mass of the Earth to be

homogeneous. If the density increase toward the center, so as to become infinite at that point, the compression instead of $\frac{1}{230}$, becomes $\frac{1}{578}$. Now the actual compression is between these limits, or about $\frac{1}{315}$; therefore the Earth is more dense toward the center than at the circumference.

It appears by Clairaut's celebrated Theorem, that if the density of the Earth increase from the surface toward the center, the compression of the spheroid, and the diminution of gravity from the pole to the equator make together a constant quantity, and are equal to $\frac{5}{2}$ of $\frac{1}{289}$.

Chap. XVIII.

ON LOGARITHMS.

(139.) All numbers in arithmetic may be expressed by the powers of 10. Thus, if $R = 10$, $R^2 = 100$, &c.; the number $5364 = 5R^3 + 3R^2 + 6R + 4$. A mixed number, as $45.678 = 4R + 5R^0 + 6R^{-1} + 7R^{-2} + 8R^{-3}$. Vulgar fractions, when transformed into decimals, may be expressed in the same way. Thus, $\frac{2}{3}$ or its equal $.666$ &c. $= 6R^{-1} + 6R^{-2} + 6R^{-3} + $ &c.

We may also express all numbers, as near as we please, by a single power of any positive number whatever, except unity. Let the numbers 2 and 10 be taken for examples; then 1, 2, 3, &c. may be expressed by the powers of these numbers.

Thus, $1 = 2^0$ $1 = 10^0$

           $2 = 2^1$ $2 = 10^{.30103, \text{&c.}}$

           $3 = 2^{1.58496, \text{&c.}}$ $3 = 10^{.47712, \text{&c.}}$

           $4 = 2^2$ &c.

           &c. $10 = 10^1$.

Hence, if r be assumed some determinate number, n an indefinite positive number, some other number N may always be found such that $r^N = n$. In every case of this kind, N is called the Logarithm of n; the logarithms, which are derived by giving a determinate value to r, constitute a system of logarithms, and r is the base of that system.

Cor. If e represent a number, whose logarithm is 1, and x be the logarithm of some other number as b, $e^x = b$.

(140.) In every system of logarithms, the logarithm of 1 must $= 0$. For if $r^N = n$, and n be assumed $= 1$, N must $= 0$; for $r^0 = 1$.

In every system the logarithm of the base is unity. For let $r = n$; then N must be unity.

The logarithms of any two given numbers have the same ratio in every system. For let $r^N = n$, and $r^B = b$; then $r^{NB} = n^B$, and $r^{NB} = b^N$; $\therefore$ $n^B = b^N$, and $n = b^{\frac{N}{B}}$. Now r is not found in this equation; therefore the value of $\frac{N}{B}$ depends only on the numbers n and b.

In every system the base of which is greater than unity, the logarithm of a whole or mixed number is positive.

If not, let $n = r^{-N}$, that is, $= \frac{1}{r^N}$; then since $\frac{1}{r^N}$ is a proper fraction, n must be a fraction; which is impossible, for n is by hypothesis greater than unity; therefore $n = r^N$.

In every system where the base is greater than unity, the logarithm of a proper fraction is negative.

If not, let $n = r^N$, where N is positive; since r^N is greater than 1, n is greater than 1; but it is by hypothesis a proper fraction, which is impossible; therefore $n = r^{-N}$.

(141.) Logarithms are also considered as measures of ratios. Thus the ratio of 81 to 3, may be considered as made up of the ratios of 81 to 27, of 27 to 9, and of 9 to 3 ; which three ratios are equal to each other, and the ratio of 81 to 3 is said to be triple the ratio of 9 to 3. In the same manner, the ratio of 100 to 1 is twice the ratio of 10 to 1 ; of 1000 to 1, three times that ratio, &c. And if the numbers A, B, C, D, are continued proportional, the ratio of $A : B$ being equal to that of $B : C$, &c., then the ratio of $A : D$ is considered as made up of three equal ratios. Hence, ratios may

be compared in respect to magnitude; thus, if two ratios can be resolved, one into 5 equal ratios, and the other into '8 of the same ratio, the magnitude of one ratio : the magnitude of the other :: 5 : 8.

In BRIGGS' system, the measure of the ratio of 10 : 1, or the logarithm of 10 is unity. NAPIER took the number 2.718282; hence, in the two systems, the logarithms of the ratios are expressed by different numbers. In a given system, the measure of the same ratio is obviously the same; of double the ratio, the measure is double, &c. A ratio of equality has no magnitude; for it has no effect in addition or subtraction; a ratio *majoris inæqualitatis* compounded with another increases it; a ratio *minoris inæqualitatis* diminishes it. If then the measure of the ratio which a greater term bears to a less be positive, the measure of the ratio which a less term bears to a greater is negative; and the measure of a ratio of equality = 0.

When this expression, the logarithm of $1 + x$, is used, it denotes the measure of the ratio of $1 + x$: 1. The logarithm of $\frac{A}{B}$, is the excess of the logarithm of A : 1 above that of B : 1, or is the measure of the ratio of A : B.

(142.) These being the principles which apply to logarithms in general, the following Propositions are intended to shew the rules by which the logarithm corresponding to any number, or the number corresponding to any logarithm, may be deduced, the one from the other. The quantity m is called the modulus (Art. 41.); it is the measure of some given ratio, and serves as a standard to which other measures may be referred.

PROBLEMS.

(143.) PROB. 1. A number being given, it is required to determine its logarithm.

Let $1+x$ be the number, y its logarithm, and m the modulus; then (Art. 41.) $\dot{y}=\dfrac{m\dot{x}}{1+x}=m(\dot{x}-x\dot{x}+x^2\dot{x}-x^3\dot{x}$

$+$ &c.) by actual division; $\therefore y = m\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\right.$

$\left.\frac{1}{4}x^4+\text{&c.}\right)$, which needs no correction; because, if $x=0$, $y=0$; for the number $1+x$ becomes 1, and the logarithm of 1 is 0.

(144.) This proposition is the same with the first in Cotes' *Harmonia Mensurarum*. In the following illustration, AB is assumed $=1$, $BC=x$, and m the modulus; BP is a variable part of BC, and $PQ=\dot{x}$.

To find the measure of any proposed ratio.

Let it be required to determine the measure of the ratio between AC and AB (of $1+x$: 1). Let the difference BC be conceived to be divided into innumerable very small parts as PQ ($\dot{x}$); then the ratio between AC and AB will be divided into as many very small ratios between AQ and AP; and if the magnitude of the ratio between AQ and AP be given, by division the ratio of $PQ : AP$ is also given, and therefore the given magnitude of the ratio between AQ and AP, may be expressed by the given quantity $\dfrac{PQ}{AP}$ (that is, $\dot{y}=\dfrac{\dot{x}}{1+x}$, if $m=1$).

If AP remain the same, and PQ be supposed to be increased or diminished in any proportion, the ratio of AQ to AP will be increased or diminished in the same proportion; thus, if PQ be doubled, tripled, &c., or if its value be changed to one half, or three halves of its former value, the ratio will become the duplicate or triplicate, the sub-duplicate or sub-triplicate; it may

therefore be still expressed by the quantity $\frac{PQ}{AP}$ (*i. e.* if

$1 + x$ be invariable, and $\dot{x}$ be doubled, tripled, &c., $\dot{y}$ will be doubled, tripled, &c.); or, if we take some constant quantity m, the measure of the ratio between AQ and

AP will be expressed by the fraction $\frac{m \times PQ}{AP} \left(\dot{y} = \frac{m\dot{x}}{1+x} \right)$.

This measure will have different magnitudes, and will be accommodated to different systems, according to the value of the assumed quantity m, which may hence be called the modulus of the system. Now, since the sum of all the ratios between AQ and AP is equal to the proposed ratio of $AC : AB$, so the sum of all the measures $\frac{m \times PQ}{AP}$ (to be found by the known methods) will be

equal to the measure required; or, $y = f \cdot \frac{m\dot{x}}{1+x} = m$

$\left(x - \frac{1}{2}x^2 + \&c. \right)$

Cor. 1. The measure of any given ratio as $1 + x : 1$, where x is given, is as the modulus (m) of the system.

Cor. 2. Since the logarithm of 2 in the common system, is ,3010300; and its logarithm in the hyperbolic is ,6931472, we have ,6931472 : ,3010300 :: 1 (the modulus in the hyperbolic system) : m, the modulus in the common system; ∴ $m = ,43424948$.

Cor. 3. Since the hyperbolic logarithm : the common logarithm :: 1 : m (,43424 &c.), the hyperbolic logarithm $= \dfrac{\text{the common logarithm}}{m}$; and the common logarithm $= m \times$ the hyperbolic.

Cor. 4. Since m may be assumed of any value, we

may, to the same number, have as many different systems
of logarithms as we please.

CoR. 5. Since y the measure of the given ratio $1+x:1$,
varies as the modulus m, $\dfrac{y}{m}$ is constant. That ratio
whose measure is m, is called by CoTes the Modular
Ratio.

(145.) By Article 143, $y=m\left(x-\dfrac{1}{2}x^2+\dfrac{1}{3}x^3-\dfrac{1}{4}x^4+\&c.\right)$
Now the smaller x is assumed, the quicker will this
series converge. If $x=1$, $y=m\left(1-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+\&c.\right)=$
the logarithm of 2 in a system whose modulus is m. If
$m=1$, $y=1-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+$ &c. = the hyperbolic loga-
rithm of 2.

PROB. 2.

(146.) To determine the measure of the ratio between
$z+x$ and $z-x$, where z is a constant and x a variable
quantity; or to find the logarithm of $\dfrac{z+x}{z-x}$.

Let y be the measure required, m the modulus; then
$$\dot{y}=2m\times\frac{z\,\dot{x}}{z^2-x^2}=2m\times\frac{\dot{x}}{z-\dfrac{x^2}{z}}=2m\times\left(\frac{\dot{x}}{z}+\frac{x^2\dot{x}}{z^3}+\frac{x^4\dot{x}}{z^5}+\&c.\right);$$

$\therefore$ the measure $y=2m\times\left(\dfrac{x}{z}+\dfrac{x^3}{3\,z^3}+\dfrac{x^5}{5\,z^5}+\&c.\right)$ CoTes,
Schol. 1.

CoR. 1. If the sum of two quantities is z, and their
difference x, and we assume $2M\times\dfrac{x}{z}=A$, $A\times\dfrac{x^2}{z^2}=B$, $\dfrac{Bx^2}{z^2}$
$=C$, $\dfrac{Cx^2}{z^2}=D$, &c., the measure of the ratio of the former

quantity to the latter $= A + \frac{1}{3} B + \frac{1}{5} C + \frac{1}{7} D + \&c.$

COR. 2. If $z = 1$, $y = 2m \left(x + \frac{1}{3} x^3 + \frac{1}{5} x^5 + \&c. \right) =$ the measure of the ratio of $1 + x : 1 - x$, or the logarithm of $\dfrac{1 + x}{1 - x}$ to a modulus m.

COR. 3. If m also $= 1$, $y = 2 \left(x + \frac{1}{3} x^3 + \frac{1}{5} x^5 + \&c. \right) =$ the hyperbolic logarithm of $\dfrac{1 + x}{1 - x}$. Let $x = \frac{1}{3}$, then $\dfrac{1 + x}{1 - x} = 2$; and y the hyperbolic logarithm of $2 = 2 \times \; : \frac{1}{3} + \frac{1}{3} \times \frac{1}{3^3} + \frac{1}{5} \times \frac{1}{3^5} + \&c.$, of which the value deduced from the first 7 terms of the series $= 0{,}6931472$. This series converges much quicker than that contained in Art. 145.

PROB. 3.

(147.) Given a logarithm, to find its number.

Let $1 + x$ be the number, and y its logarithm; then $\dot{y} = \dfrac{m \dot{x}}{1 + x}$; $\therefore \dot{y} + x \dot{y} = m \dot{x}$, and $\dot{y} + x \dot{y} - m \dot{x} = 0$. Assume a series $ay + by^2 + cy^3 + dy^4 + \&c. = x$; then $\dot{x} = a \dot{y} + 2by \dot{y} + 3cy^2 \dot{y} + \&c.$; $\therefore$ by substituting in the equation $\dot{y} + x \dot{y} - m \dot{x} = 0$, these values of x and of $\dot{x}$, we have

$$\left. \begin{array}{l} \dot{y} + ay\dot{y} + by^2 \dot{y} + \&c. \\ - ma \dot{y} - 2mby \dot{y} - 3mcy^2 \dot{y} - \&c. \end{array} \right\} = 0 \; ;$$

hence, $1 - ma = 0$, or $a = \dfrac{1}{m}$; $a - 2mb = 0$, or $b = \dfrac{a}{2m} = \dfrac{1}{2m^2}$; $b = 3mc = 0$, and $c = \dfrac{1}{2.3.m^3}$ &c.; $\therefore x = \dfrac{y}{m} + \dfrac{y^2}{2m^2} + \dfrac{y^3}{2.3.m^3} + \&c.$, and $1 + x = 1 + \dfrac{y}{m} + \dfrac{y^2}{2m^2} + \dfrac{y^3}{2.3.m^3} + \&c. =$ the number whose logarithm is y.

Cor. If $m=1$, we obtain the number whose hyperbolic logarithm is y; it $= 1 + y + \dfrac{y^2}{3} + \dfrac{y^3}{2 \cdot 3} + \&c.$

PROB. 4.

(148.) To find the modular ratio.

By the last Proposition, the number whose logarithm is

$$y = 1 + \frac{y}{m} + \frac{y^2}{2 \, m^2} + \frac{y^3}{2 \cdot 3 \cdot m^3} + \&c.; \text{ or } y \text{ is the measure}$$

of the ratio of this number to 1. Now the modular ratio is that ratio of which the modulus is the measure; if then $m = y$, m will be the measure of this ratio, and the ratio itself becomes the modular ratio; that is, the modular

ratio is the ratio of $1 + 1 + \dfrac{1}{2} + \dfrac{1}{2 \cdot 3} + \&c.$: 1; and

therefore is the same for every system of logarithms, being independent both of m and y.

(149.) By summing the series, it appears that this modular ratio is that of 2,7182818 &c. : 1. In Napier's system, where the modulus is 1, the logarithm of 2,7182818 is 1. The measure of this same ratio will be the modulus in any other system; the measure of that ratio in Briggs' system (where the logarithm of $10 = 1$), is ,43429448, the modulus of his system.

(150.) In Art. 143. we assumed y the measure of the ratio of $1 + x$: 1, or the logarithm of $1 + x$. If y be the

logarithm of $\dfrac{1}{1 - x}$, or $-y$ the logarithm of $1 - x$, then

$$-\dot{y} = \frac{-m\dot{x}}{1 - x} \text{ and } \dot{y} = \frac{m \, \dot{x}}{1 - x} = m \times \; : \dot{x} + x\dot{x} + x^2\dot{x} + x^3\dot{x} + \&c.;$$

and $y = m \times \; : x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \&c.$

(151.) In the same manner, if in Art. 147. we take y

L

the logarithm of $\dfrac{1}{1-x}$, $\dot{y} - x\dot{y} - m\dot{x} = 0$; and by assuming

$x = ay + by^2 + cy^3 + \&c.$ as before, we get $1 - x = 1 - \dfrac{y}{m} +$

$\dfrac{y^2}{2\,m^2} - \dfrac{y^3}{2\,.\,3\,m^3} + \&c.$

Cor. 1.   The modular ratio in this case = the ratio

of $1 : 1 - \dfrac{1}{1} + \dfrac{1}{2} - \dfrac{1}{2\,.\,3} + \&c.$, or of $1 : ,3678794$ &c.

Cor. 2.   Since $1 + \dfrac{1}{1} + \dfrac{1}{2} + \dfrac{1}{2\,.\,3} + \&c. : 1 :: R + \dfrac{R}{1} +$

$\dfrac{R}{2} + \dfrac{R}{2\,.\,3} + : R$, if $\dfrac{R}{1}$ be assumed $= A$, $\dfrac{A}{2} = B$; $\dfrac{B}{3} = C$;

$\dfrac{C}{4} = D$, &c. *ad infinitum*, and S be put equal to $R + A +$

$B + C + D + \&c.$ the modular ratio will be that of $S : R$.

Or, since $1 : 1 - \dfrac{1}{1} + \dfrac{1}{2} - \dfrac{1}{2\,.\,3} + \&c. :: S : S - \dfrac{S}{1} +$

$\dfrac{S}{2} - \dfrac{S}{2\,.\,3} + \&c.$ *ad infinitum*, if $\dfrac{S}{1} = A$, $\dfrac{A}{2} = B$, $\dfrac{B}{3} = C$, &c.

and R be taken equal to $S - A + B - C + \&c.$ *ad infinitum*,
the modular ratio is that of $S : R$.   By the preceding
cases, this ratio equal that of $2,7182818$ &c. $:: 1$, or of $1 :$
$,3678794$, &c.

CHAP. XIX.

ON THE APPLICATION OF FLUXIONS TO THE MOTIONS OF BODIES AFFECTED BY CENTRIPETAL FORCES.

SECTION I.

PROPOSITION.

(152.) To deduce a fluxional expression, for the variation of the force in any given curve, considered as a spiral.

If S be the center of force corresponding to any orbit AB, Pv the chord of curvature at P, and Sy a perpendicular upon the tangent Py, the force by which a body would revolve in the orbit AB varies as $\dfrac{1}{Sy^2 \times Pv}$.

(NEWTON, Sect. 2. Prop. 6.)
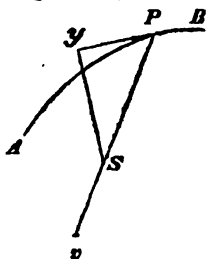
Let $SP = x$, $Sy = p$; then $Pv = \dfrac{2p\dot{x}}{\dot{p}}$ (Art. 118.) ;

∴ by substitution $F \propto \dfrac{1}{p^2 \times \dfrac{2p\dot{x}}{\dot{p}}} \propto \dfrac{\dot{p}}{p^3\dot{x}}$.

EXAMPLES.

(153.) Ex. 1. To find the law of force, by which a body may be made to describe an ellipse round the center.

Let AC the semi-axis major $= a$, CB the semi-axis minor $= b$, $CP = x$, CD the semi-conjugate $= y$, and Cy the perpendicular on the tangent $= PF = p$.
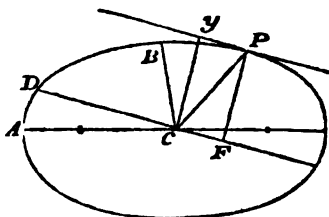
Then, by the nature of the curve, $PF^2 \times CD^2 = AC^2 \times BC^2$, or $\frac{1}{p^2} = \frac{y^2}{a^2 b^2}$; $\therefore -\frac{\dot{p}}{p^3}$ varies as $y\dot{y}$. Also

$x^2 + y^2 = a^2 + b^2$; $\therefore x\dot{x} = -y\dot{y}$; $\therefore \frac{\dot{p}}{p^3}$ varies as $x\dot{x}$,

and $\frac{\dot{p}}{p^3\dot{x}}$ varies as x, or the force varies as the distance from the center.

Cor. If the ellipse be changed into a parabola by the indefinite production of the axis major, then at any finite distance from the vertex A, x is infinite; hence in this case the force is constant, and may be conceived to act in a direction parallel to the axis.

Ex. 2. To find the law of the force, by which a body may be made to describe an hyperbola; the center of force being in the center of the figure.

Here $PF^2 \times CD^2 = CA^2 \times CB^2$, or $\frac{1}{p^2} = \frac{y^2}{a^2 b^2}$; $\therefore \frac{-\dot{p}}{p^3}$ varies

as $y\dot{y}$, and $\frac{\dot{p}}{p^3\dot{x}} \propto \frac{-y\dot{y}}{\dot{x}}$. Also $x^2 - y^2 = a^2 - b^2$; $\therefore x\dot{x} = y\dot{y}$;

$\therefore \frac{\dot{p}}{p^3\dot{x}}$ varies as $-x$; or the
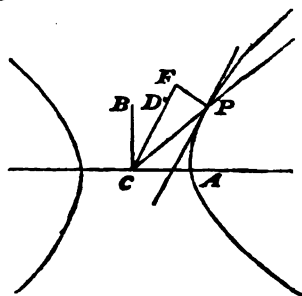
force is repulsive, and proportional to the distance from the center.

Ex. 3. To find the law of the force in a parabola round the focus S.

Let $SP = x$, $Sy = p$, $SA = a$; then

$p^2 = ax$, or varies as x; $\therefore \dfrac{1}{p^2}$ is as $\dfrac{1}{x}$,

and $\dfrac{\dot{p}}{p^3}$ as $\dfrac{\dot{x}}{x^2}$; $\therefore \dfrac{\dot{p}}{p^3\dot{x}}$ varies as $\dfrac{1}{x^2}$, or the force is inversely as SP^2.
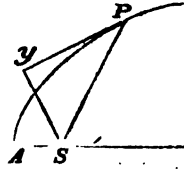
Ex. 4. To find the same in an ellipse round the focus.

Let $AC = a$, $CB = b$, $SP = x$, $Sy = p$, $HP = v$. Then, by Conic Sections, $Sy^2 =$

$\dfrac{b^2 \times x}{v}$; $\therefore \dfrac{1}{p^2}$ varies as $\dfrac{v}{x}$, and

$\dfrac{-\dot{p}}{p^3}$ is as $\dfrac{x\dot{v} - v\dot{x}}{x^2}$. But $x + v$
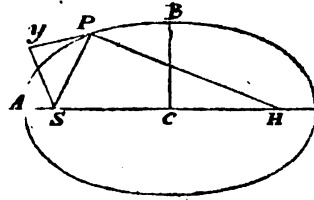
$= 2a$, $\therefore \dot{v} = -\dot{x}$; and $\dfrac{-\dot{p}}{p^3}$

varies as $-\dfrac{x\dot{x} + v\dot{x}}{x^2}$; $\therefore \dfrac{\dot{p}}{p^3\dot{x}}$ varies as $\dfrac{x + v}{x^2}$, or as $\dfrac{2a}{x^2}$.

Hence the force is as $\dfrac{1}{SP^2}$.

Ex. 5. To find the same in the hyperbola round the focus.

Assuming as before, $p^2 = \dfrac{b^2 x}{v}$; $\therefore \dfrac{1}{p^2}$ varies as $\dfrac{v}{x}$, and

$\dfrac{-\dot{p}}{p^3}$ varies as $\dfrac{\dot{v}x - \dot{x}v}{x^2}$.

But $v - x = 2a$, and $\dot{v} = \dot{x}$;

$\therefore \dfrac{-\dot{p}}{p^3}$ is as $\dfrac{x\dot{x} - v\dot{x}}{x^2}$, or as

$-\dot{x} \times \dfrac{v - x}{x^2}$; $\therefore \dfrac{\dot{p}}{p^3\dot{x}}$

varies as $\dfrac{v - x}{x^2}$, or as $\dfrac{2a}{x^2}$;
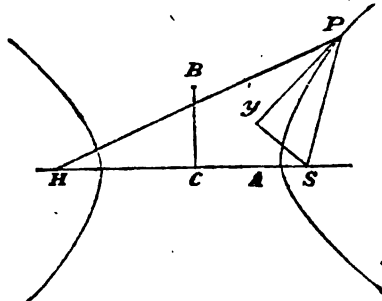
that is, the force varies inversely as SP^2.

Ex. 6. To find the same, if the center of force be situate in the pole of a logarithmic spiral.

Here $SP : Sy :: a : b$, or x varies as p; therefore $\dfrac{1}{p^3}$ varies as $\dfrac{1}{x^3}$, and $\dfrac{\dot{p}}{p^3\dot{x}}$ as $\dfrac{1}{x^3}$; or the force varies inversely as SP^3.
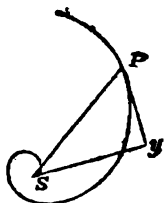
Ex. 7. If the center of force be situated in a point S, which is not the center of the circle; to find the law of the force.

Suppose the body at P; join PS, and produce it to v; draw vA a diameter, Py a tangent, and Sy a perpendicular on the tangent; join AP. Let $Av = a$, $SP = x$, $Sy = p$.

By similar triangles, AvP, SPy, $SP : Sy :: Av : Pv$; $\therefore Sy$ varies as $SP \times Pv$, or p varies as $x \times \dfrac{2p\dot{x}}{\dot{p}}$; $\therefore \dfrac{\dot{p}}{\dot{x}}$ varies as x, and $\dfrac{\dot{p}}{p^3\dot{x}}$ as $x \times \dfrac{1}{x^3 \times Pv^3}$; or the force varies as $\dfrac{1}{SP^2 \times Pv^3}$.
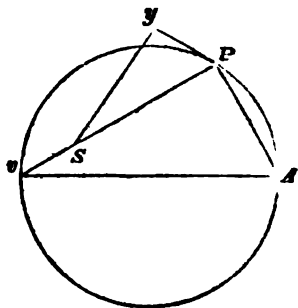
Cor. If the point S coincide with v, or the center of force be in the circumference, $F \propto \dfrac{1}{Pv^5}$.

Ex. 8. To find the law of the force, by which a body may describe the semi-circle APB, the force acting in parallel lines, and perpendicular to AB.

Draw PM perpendicular to AB, and suppose it produced indefinitely to S; draw PY a tangent, and let SY be conceived perpendicular to PY; join CP. Then,

by similar triangles, CPM, SPY, $SP^2 : SY^2 :: CP^2 : PM^2$; $\therefore SY^2$ varies as PM^2, since SP^2 is infinite; or, if $SY=p$, and $PM=y$,

$$\frac{1}{p^2} \propto \frac{1}{y^2}; \quad \therefore \frac{\dot{p}}{p^3} \propto \frac{\dot{y}}{y^3}; \text{ but}$$

$\dot{y} =$ the fluxion of $SP = \dot{x}$;

$\therefore \dfrac{\dot{p}}{p^3\dot{x}}$ varies as $\dfrac{1}{y^3}$, or the

force varies as $\dfrac{1}{PM^3}$.
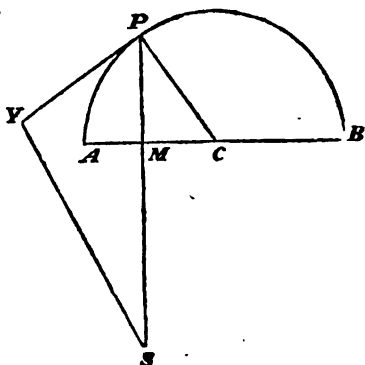
Ex. 9. Let APB be a cycloid, and the force act in parallel lines perpendicular to the base AB.

Let V be the vertex, and VDC the generating circle.
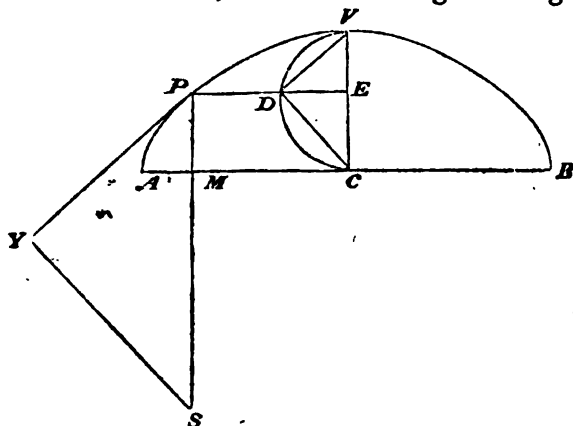
Then if PDE be drawn parallel to AB, and DV, DC, be joined, $SP : SY :: CV : CD :: \sqrt{CV} : \sqrt{CE}$;

$\therefore SY \propto \sqrt{CE} \propto \sqrt{y}$; hence $\dfrac{1}{p^2} \propto \dfrac{1}{y}$, and $\dfrac{\dot{p}}{p^3} \propto \dfrac{\dot{y}}{y^2} \propto \dfrac{\dot{x}}{y^2}$;

$\therefore \dfrac{\dot{p}}{p^3\dot{x}} \propto \dfrac{1}{y^2}$, and the force $\propto \dfrac{1}{PM^2}$.

Ex. 10. To find the law of the force in a spiral, whose

equation is $p = \dfrac{ax}{\sqrt{a^2+x^2}}$.

Here $\dfrac{1}{p^2} \propto \dfrac{a^2 + x^2}{x^2}$; $\therefore -\dfrac{2\dot{p}}{p^3} \propto \dfrac{2x^3\dot{x} - 2x\dot{x}(a^2 + x^2)}{x^4} \propto$

$-\dfrac{2a^2 x\dot{x}}{x^4}$; $\therefore$ the force, or $\dfrac{\dot{p}}{p^3\dot{x}}$, $\propto \dfrac{1}{x^3}$.

Ex. 11. Let the curve be the involute of a circle, and the center of force in the center of the circle; to find the law.

Take $SP = x$, $Sy = p$; then, by the nature of the curve, $Py = r$, and

$p = \sqrt{x^2 - r^2}$; $\therefore \dfrac{1}{p^2} = \dfrac{1}{x^2 - r^2}$; and

$\dfrac{\dot{p}}{p^3\dot{x}} \propto \dfrac{x}{(x^2 - r^2)^2} \propto \dfrac{SP}{Sy^4}$.
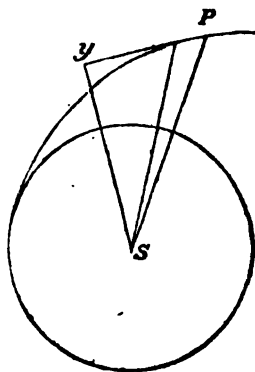
SECTION II.

(154.) *On the motion of bodies in non-resisting mediums, when urged by forces tending to a center, aud varying according to some power of the distance.*

PROPOSITION.

(155.) If v represent a velocity generated by a constant force F through a space x, $v^2 = 4mFx$. Let $AB =$ the space through which the body has descended; let $BD = \dot{x}$, and F represent the force at B. Then however F may vary, yet through a small element of space BD, it may be considered as constant; therefore $2v\dot{v} = +4mF\dot{x}$, and $v\dot{v} = +2mF\dot{x}$. If $CB = x$, then $BD = -\dot{x}$; therefore in this case $2v\dot{v} = -4mF\dot{x}$; $\therefore v\dot{v} = -2mF\dot{x}$, whence in general $v\dot{v} = \pm 2mF\dot{x}$.

Cor. $v\dot{v}$ varies as $\pm F\dot{x}$.

If $F=1$, and z represent the space through which a body must fall by this force to acquire the velocity v, $v^2=4mz$; $\therefore v\dot{v}=\pm 2m\dot{z}$;

$\qquad \therefore 2m\dot{z} = \pm 2mF\dot{x}$, and $\dot{z} = \pm F\dot{x}$.

If x decrease as the velocity increases, $\dot{z}=-F\dot{x}$. If x increases with the increase of velocity, $\dot{z} = +F\dot{x}$; and $v = \sqrt{4mz} = \sqrt{4m} \times \sqrt{f \cdot F\dot{x}}$.

PROPOSITION.

(156.) Let $T=$ the time in which a body acquires the velocity v, when urged by the constant force F; then $v = 2mFT$; $\therefore$ for a small element of time as $\dot{T}$, $\dot{v} = 2mF\dot{T}$. But

$$S \text{ or } x = \frac{T \times V}{2}; \therefore \pm\dot{x} = \frac{T\dot{v} + v\dot{T}}{2} = \frac{2mFT\dot{T} + 2mFT\dot{T}}{2}$$

$$= 2mFT\dot{T} = V \times \dot{T}; \therefore \dot{T} = \frac{\pm\dot{x}}{V}.$$ If x increase with

the increase of the time, $\dot{T} = \frac{+\dot{x}}{v}$; if x decrease as the

time increases, $\dot{T} = \frac{-\dot{x}}{V}$.

EXAMPLES.

(157.) Ex. 1. Suppose the force to vary as the distance^{n-1} from the center; to find the variation of the velocity.

Here $v\dot{v} \propto -F\dot{x} \propto -x^{n-1}\dot{x}$; $\therefore v^2 \propto \dfrac{-x^n}{n} + \text{corr.}$ Let

$v=0$, when $x=a$; then $v^2 \propto a^n - x^n$, and $v \propto (a^n - x^n)^{\frac{1}{2}}$.

Ex. 2. To find the variation of the time.

Here $\dot{T} \propto \dfrac{-\dot{x}}{(a^n - x^n)^{\frac{1}{2}}}$, and the fluent gives the variation

of T; but this fluent can only be found in particular cases.

Ex. 3.   Let the force $\propto \dfrac{1}{D^{n+1}}$, where D is the distance from the center; to find the variation of the velocity.

Here $v\dot{v}\propto \dfrac{-\dot{x}}{x^{n+1}}$; $\therefore v^2\propto \dfrac{1}{nx^n}+$ corr.   Let $v=0$, when $x=a$; then $v^2\propto \left(\dfrac{1}{x^n}-\dfrac{1}{a^n}\right)\propto \dfrac{a^n-x^n}{x^n}$; $\therefore v\propto \left(\dfrac{a^n-x^n}{x^n}\right)^{\frac{1}{2}}$.

Ex. 4.   To find the variation of the time.

Here $T\propto \dfrac{-x^{\frac{n}{2}}\dot{x}}{(a^n-x^n)^{\frac{1}{2}}}$; of which the fluent may be found in particular cases.

Ex. 5.   To find the actual velocity acquired by a body in falling through a space AB, when urged by a constant force F, situated in the point C.

Let $AC=a$, and $x=$ any variable distance; then $v\dot{v}=-2mF\dot{x}$, and $\dfrac{v^2}{2}=-2mFx+$cor. When $v=0$, $x=a$; $\therefore v^2=4mF(a-x)$, and $v=\left(4mF(a-x)\right)^{\frac{1}{2}}$.

Cor.   The velocity acquired through $AB\propto \sqrt{AB}$.

Ex. 6.   To find the time of falling.

Here $T=\dfrac{-\dot{x}}{V}=\dfrac{-\dot{x}}{(4mF)^{\frac{1}{2}}.(a-x)^{\frac{1}{2}}}=\dfrac{1}{(4mF)^{\frac{1}{2}}}-\dot{x}(a-x)^{-\frac{1}{2}}$,

$\therefore T=\dfrac{2}{(4mF)^{\frac{1}{2}}}(a-x)^{\frac{1}{2}}+$corr.; and when $x=a$, $T=0$;

$\therefore$ corr.$=0$.   Hence, $T=\left(\dfrac{1}{mF}.(a-x)\right)^{\frac{1}{2}}$.

Cor. The time $\propto \sqrt{AB}$.

Ex. 7. Let a body fall from rest at A, when urged by a force situated in C, which varies directly as the distance; to find the velocity acquired through AB.

Let the force at some point $M = F$; then, if $CM = r$, and $x =$ any variable distance, $r : x :: F :$ the force at $B = \dfrac{Fx}{r}$; $\therefore v\dot{v} = -\dfrac{2mF}{r} \times x\dot{x}$, and $v^2 = \dfrac{2mF}{r} \times -x^2 + \text{corr.} = \dfrac{2mF}{r}(a^2 - x^2)$; $\therefore$ the velocity $= \left(\dfrac{2mF}{r} \cdot (a^2 - x^2)\right)^{\frac{1}{2}}$.

Cor. 1. If with C as a center, and CA radius, the quadrant of a circle be described, and BD be drawn perpendicular to AC, the velocity at $B \propto BD$, $\propto$ the right sine of a circular arc, whose radius is the greatest distance from which the body begins to fall, and versed sine is the space through which it has descended.
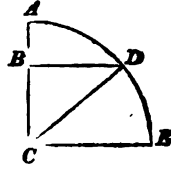
Cor. 2. The velocity acquired at B : that acquired at C :: $BD : CE$ :: $BD : CD$.

Cor. 3. Since the force in a cycloid varies as the distance of the pendulum from the lowest point, if a pendulum be conceived to oscillate in a semi-cycloid, $CL =$ the line CA, and the magnitude of the force at $L =$ that at A, its motion will correspond with that of a body descending through AC; and if $Cb = CB$, the velocity of the pendulum at b must be as the right sine of a circular arc, whose radius $= CL$, and versed sine $=$ the space Lb.
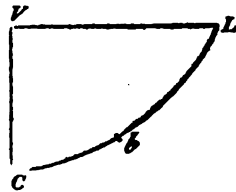
Ex. 8. The same things remaining, let it be required to find the time.

Here $\dot{T} = \dfrac{-\dot{x}}{V} = \left(\dfrac{r}{2mF}\right)^{\frac{1}{2}} \cdot \dfrac{-\dot{x}}{\sqrt{a^2 - x^2}} =$

$$\left(\frac{r}{2\,m\,Fa^2}\right)^{\frac{1}{2}}\cdot\frac{-a\dot{x}}{\sqrt{a^2-x^2}}\;;\;\therefore\;T=\left(\frac{r}{2\,m\,Fa^2}\right)^{\frac{1}{2}}.\;\text{a circular}$$

arc, whose radius $=a$, and cosine $x=\left(\dfrac{r}{2\,m\,Fa^2}\right)^{\frac{1}{2}}$. the arc

AD. (Art. 44. Case 5. Cor. 5.)

Cor. 1. The time $\propto$ as the arc AD; and T, AB ::
T, AC :: AD : AE.

Cor. 2. Since the arc of the quadrant $AE=\dfrac{pa}{2}$, where

$p=3.14159$, &c., the actual time of descent from A

to $C=\left(\dfrac{r}{2mFa^2}\right)^{\frac{1}{2}}\cdot\dfrac{pa}{2}=\dfrac{p}{2}\left(\dfrac{r}{2\,m\,F}\right)^{\frac{1}{2}}$.

Cor. 3. In this expression a does not appear; hence, when the force varies as the distance from the lowest point, the times of descent from all altitudes to that point are the same, whether the bodies descend in straight lines, or in curves.

Cor. 4. The time of descent from A to $C=$ half the time in which a pendulum, whose length is r, would oscillate by the action of the constant force F.

Ex. 9. Let the force vary inversely as the distance from C, and be repulsive; to find the velocity acquired through a given space AB, and the time of describing it.

Let the force at some point $F=F$; then if $CD=x$,
$CF=r$, the force at $D=\dfrac{Fr}{x}$, and $v\dot{v}=+\dfrac{2\dot{m}Fr\dot{x}}{x}$;

$\therefore v^2=4mFr\times$ hyp. log. $x+$corr. Let $CA=a$, the velocity being nothing at A; then $v=$
$\left(4mFr\times\text{hyp. log.}\dfrac{x}{a}\right)^{\frac{1}{2}}$.

Also $T = + \dfrac{\dot{x}}{v} = \dfrac{1}{(4\,m\,Fr)^{\frac{1}{2}}} \times \dfrac{\dot{x}}{\left(\text{hyp. log.}\ \dfrac{x}{a}\right)^{\frac{1}{2}}}$, and

$T = \dfrac{1}{(4\,m\,Fr)^{\frac{1}{2}}} : 2av^{\frac{1}{2}} + \dfrac{2\,a\,v^{\frac{3}{2}}}{3\,.\,1} + \&c.$, where $v = $ hyp.

log. of $\dfrac{x}{a}$ (*fl.* 130.)

Ex. 10. Let the force vary as $\dfrac{1}{\overline{\text{dist.}}\rvert^2}$ from C; to find

the velocity and time.

Here the force at a variable distance $x = \dfrac{Fr^2}{x^2}$; $\therefore v\dot{v} =$

$-2mFr^2 \times \dfrac{\dot{x}}{x^2}$, and $v^2 = 4mFr^2\left(\dfrac{1}{x} - \dfrac{1}{a}\right)$ when corrected,

$= 4mFr^2 \times \dfrac{a-x}{ax}$; $\therefore v = \left(\dfrac{4\,mFr^2}{a}\right)^{\frac{1}{2}} \cdot \left(\dfrac{a-x}{x}\right)^{\frac{1}{2}}$.

Also $\dot{T} = \dfrac{-\dot{x}}{v} = \left(\dfrac{a}{4mFr^2}\right)^{\frac{1}{2}} \cdot \dfrac{-x^{\frac{1}{2}}\dot{x}}{\sqrt{a-x}} = \left(\dfrac{a}{4mFr^2}\right)^{\frac{1}{2}} \cdot$

$\dfrac{-x\dot{x}}{(ax-x^2)^{\frac{1}{2}}}$; and $T = \left(\dfrac{a}{4mFr^2}\right)^{\frac{1}{2}} : (ax-x^2)^{\frac{1}{2}}$

— a circular arc of radius $\frac{1}{2}\,a$, and versed sine x (*fl.* 23.); that is, if upon AC as a diameter a semi-circle be described, and the ordinate BE be drawn, the time through
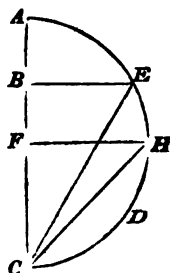
$AB = \left(\dfrac{a}{4\,mFr^2}\right)^{\frac{1}{2}} : BE - CDE + \text{corr.} = \left(\dfrac{a}{4\,mFr^2}\right)^{\frac{1}{2}} :$

$BE - CDE + CDA = \left(\dfrac{a}{4mFr^2}\right)^{\frac{1}{2}} : BE + AE.$

Cor. 1. The velocity at $B \propto \left(\dfrac{AB}{AC \times CB}\right)^{\frac{1}{2}}$; or in different parts of the same descent $\propto \left(\dfrac{AB}{CB}\right)^{\frac{1}{2}}$.

Cor. 2. The velocity acquired in falling to the center is infinite.

Cor. 3. The time down $AB \propto$ the arc + the sine.

Cor. 4. If $p =$ the circumference of a circle, whose diameter is 1, $\dfrac{pa}{2} = ADC$, the semi-circumference to a diameter a; and the whole time of descent from A to C

$$= \left(\frac{a}{4mFr^2}\right)^{\frac{1}{2}} \cdot \frac{pa}{2} = \frac{p}{4r} \times \frac{a^{\frac{1}{2}}}{\sqrt{mF}}.$$

Cor. 5. If Q represent the length of a quadrant, R the radius of the circle, and the force $\propto \dfrac{1}{D^2}$, the time of descent through the first half of a straight line toward the center : the time through the remaining half ::
$Q + R : Q - R$.

Bisect AC in F, and draw FH perpendicular to AC; then, T down $AF : T$, down $AC :: AH + HF : ADC$; $\therefore T, AF : T, FC :: AH + HF : ADC - AH - HF$
$$:: \quad Q + R : Q - R.$$

Ex. 11. Let the force tending to C vary inversely as the cube of the distance; to find the velocity and time.

In this case, the force at a variable distance

$x = \dfrac{Fr^3}{x^{3}}$; $\therefore v\dot{v} = -2mFr^3 \times \dfrac{\dot{x}}{x^3}$; $\therefore v^2 = 2mFr^3 \times$

$\dfrac{1}{x^2} + \text{corr.} = 2mFr^3 \times \left(\dfrac{1}{x^2} - \dfrac{1}{a^2}\right) = 2mFr^3 \times \dfrac{a^2 - x^2}{a^2 x^2}$;

$\therefore v = (2mFr^3)^{\frac{1}{2}} \times \dfrac{(a^2 - x^2)^{\frac{1}{2}}}{ax}$.

Also $\dot{T} = \dfrac{a}{(2mFr^3)^{\frac{1}{2}}} \times \dfrac{-x\dot{x}}{(a^2-x^2)^{\frac{1}{2}}}$; and $T = \dfrac{a}{(2mFr^3)^{\frac{1}{2}}}$. $(a^2-x^2)^{\frac{1}{2}}$.

Cor. 1. With C as a center, and $Ca = CA$ as radius, describe a circle aDE. Take $ab = AB$; draw bD perpendicular to aC, and let aT be the tangent of aD. Then $Cb : bD :: Ca : aT$, or $x : \sqrt{a^2 - x^2} :: a : aT$; $\therefore aT \propto \dfrac{\sqrt{a^2 - x^2}}{x}$; but $v \propto$ $\dfrac{\sqrt{a^2 - x^2}}{x}$; $\therefore$ the velocity acquired
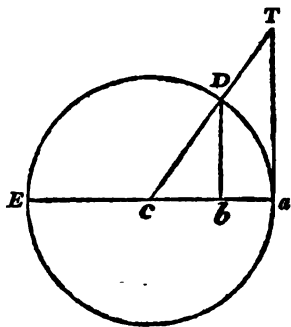
through $AB \propto$ as the tangent of a circular arc, whose radius is CA, and versed sine the space through which the body has descended.

Cor. 2. The velocity acquired at C is infinite.

Cor. 3. The time through $AB \propto Db \propto$ the right sine of the arc, whose versed sine is the space described.

Ex. 12. If the force $\propto D^{n-1}$, required an expression for the whole time of descent to the center.

In this case, the force at a distance $x = \dfrac{Fx^{n-1}}{r^{n-1}}$;

$\therefore v\dot{v} = \dfrac{-2mF}{r^{n-1}} \times x^{n-1}\dot{x}$; and $v^2 = \dfrac{4mF}{nr^{n-1}} \times (a^n - x^n)$;

$\therefore v = \left(\dfrac{4mF}{nr^{n-1}}\right)^{\frac{1}{2}} \cdot (a^n - x^n)^{\frac{1}{2}}$.

Hence $\dot{T} = \dfrac{-\dot{x}}{v} = \left(\dfrac{nr^{n-1}}{4mF}\right)^{\frac{1}{2}} \cdot \dfrac{-\dot{x}}{(a^n - x^n)^{\frac{1}{2}}} = \left(\dfrac{nr^{n-1}}{4mFa^n}\right)^{\frac{1}{2}} \cdot$

$$-\dot{x} \times \overline{1 - \frac{x^n}{a^n}}\,^{-\frac{1}{2}} = \text{(by the Binomial Theorem)} \left(\frac{nr^{n-1}}{4mFa^n}\right)^{\frac{1}{2}}$$

$$: -\dot{x} - \frac{x^n \dot{x}}{2\,a^n} - \frac{3x^{2n}\dot{x}}{2.4.a^{2n}} - \frac{3.5.x^{3n}\dot{x}}{2.4.6.a^{3n}} - \&c.;\ \therefore\ T =$$

$$\left(\frac{nr^{n-1}}{4mFa^n}\right)^{\frac{1}{2}} : -x - \frac{x^{n+1}}{2.(n+1).a^n} - \frac{3\,x^{2n+1}}{2.4.(2n+1).a^{2n}} - \&c.$$

$+$ corr. Now when $T=0$, $x=a$; and when the body reaches the center, all the terms which contain x must vanish; therefore the whole time $= \left(\dfrac{nr^{n-1}}{4mFa^n}\right)^{\frac{1}{2}} : a +$

$$\frac{a^{n+1}}{2.(n+1).a^n} + \frac{3\,a^{2n+1}}{2.4.(2n+1).a^{2n}} + \&c. = \left(\frac{nr^{n-1}}{4\,m\,F}\right)^{\frac{1}{2}} : \frac{1}{a^{\frac{n-1}{2}}}$$

$$+ \frac{1}{2.(n+1).a^{\frac{n-1}{2}}} + \frac{3}{2.4.(2n+1).a^{\frac{n-1}{2}}} + \&c.$$

COR. The whole time $\propto \dfrac{1}{a^{\frac{n-1}{2}}}$.

EXAMPLES.

1. Let the force be constant; $n-1=0$; $\therefore\ n=1$, and $T \propto \sqrt{a}$.

2. Let $F \propto$ the distance; $n - 1 = 1$; $\therefore\ n = 2$; $\therefore\ T \propto \dfrac{1}{a^0} \propto 1$, or is constant from all altitudes.

3. Let $F \propto \dfrac{1}{D^2}$; $n - 1 = -2$; $\therefore\ n = -1$; and $T \propto a^{\frac{3}{2}}$.

Ex. 13. If the attractive force in C vary as $\dfrac{1}{D^{n+1}}$, and a body be projected from B in the direction BA, with a

given velocity; required to find the height to which it will rise. (See the second FIGURE in page 175.)

Let CA, the height required, $=p$, $CB=r$, CE a variable distance $=x$; let $c=$ the velocity of projection, and v the velocity at E. Let the force at $B=F$. Then the force at $E=\dfrac{Fr^{n+1}}{x^{n+1}}$; $\therefore v\dot{v}=-\dfrac{2\,m\,Fr^{n+1}\dot{x}}{x^{n+1}}$; and $v^2=$

$\dfrac{4\,m\,Fr^{n+1}}{n\,x^n}+$ corr. When $v=c$, $x=r$; $\therefore v^2-c^2=$

$\dfrac{4\,m\,Fr^{n+1}}{n\,x^n}-\dfrac{4\,m\,Fr}{n}$. Let $v=0$, or the body arrive at

A; $x=p$, and $c^2=\dfrac{4\,m\,Fr}{n}-\dfrac{4\,m\,Fr^{n+1}}{np^n}$; $\therefore np^n c^2=$

$4\,m\,Frp^n-4\,m\,Fr^{n+1}$. Hence, $p^n(4\,m\,Fr-nc^2)=$

$4\,m\,Fr^{n+1}$; $\therefore p=r\left(\dfrac{4\,m\,Fr}{4\,m\,Fr-nc^2}\right)^{\frac{1}{n}}$.

Cor. If $F=1$, and n also $=1$, or the force $\propto\dfrac{1}{D^2}$,

$p=r\times\dfrac{4mr}{4mr-c^2}$; hence, if $c=$ the velocity acquired

through $\dfrac{r}{2}$, by the constant force at B, or $c^2=2mr$, $p=2r$.

SECTION III.

PROPOSITION.

(158.) If a body describe any curve round a center of force, the velocity at any point varies inversely as the perpendicular drawn from that center upon the tangent.

Let C be the center of force; AB, BE, ED, small

M

parts of the curve described in equal
successive portions of time. On Ey,
a tangent at E, draw the perpendi-
cular Cy; then CED may be con-
sidered as a small rectilinear triangle,
and its area $CED \propto DE \times Cy$;
$\therefore DE \propto \dfrac{CED}{Cy}$. But the arc DE
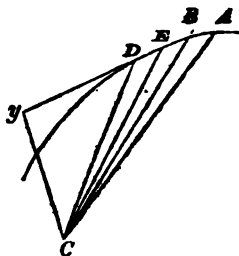
described in a small given time is proportional to the
velocity at E; and (by NEWTON, Prop. 1.) all the areas
ACB, BCE, ECD, &c. described round C, *dato tempore*,
are equal; $\therefore v \propto \dfrac{1}{Cy}$.

PROPOSITION.

(159.) Let C represent a center of force, and suppose
one body to describe round that center a curve VIK,
whilst another descends in a right line by the action of
the same force toward C; if their velocities are equal in
one instance, when at equal distances from C, in the
curve and the straight line, they will be equal at all other
equal distances.

Let IK represent a small part of the curve; with C as
a center, and radii CI, CK, describe the
circular arcs ID, KE; draw NT per-
pendicular to IK; and suppose that the
velocity of the body descending from A
to C, when at D, to equal the velocity
in the curve at I. Let DE or IN re-
present the force at D or I toward the
center; IT will represent that part of
it which impels the body in the curve.
Now the times in which DE and IK are described, since
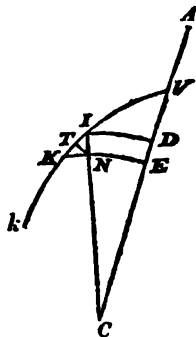
the velocities are equal at D and I, will be proportional to DE and IK; and the increment of the velocity is as the force and the increment of the time, or $\dot{v} \propto F \times \dot{T}$;

But F in the curve : F in the line :: IT : DE,
and T in the curve through IK : T through DE :: IK : DE.

Hence $(F \times \dot{T}$ or$)$ $\dot{v}$ in curve : $(f \cdot \times \dot{t})$ or $\dot{v}$ in line :: $IT \times IK : DE^2$.
But since INK is a right angle, $IT \times IK = IN^2 = DE^2$; $\therefore$ the increments of velocity through IK and DE are equal. In the same manner, the corresponding increments, and therefore the whole velocities at equal distances, will continue to be equal.

Cor. 1. If the body at D projected upward with the velocity at D would ascend to A before it loses all its velocity, the body at I, projected from I in the direction CI produced, would rise to a distance equal to CA.

Cor. 2. Hence, since $v\dot{v} \propto \pm F\dot{x}$ in the line, the same expression obtains in the curve.

PROPOSITION.

(160.) If a body revolve uniformly in a circle round the center C, the velocity of the body is equal to that which it would acquire by falling through one-fourth of the diameter, when urged by the constant force in the circumference.

Take AD a tangent at A, very small. Draw BD parallel to AC, and BE perpendicular to it; and suppose Ax to be the space through which a body must fall to acquire the velocity at A. Now whilst the body describes AB with an uniform velocity, another would fall through DB by the constant force at A, and would afterwards describe $2DB$ in the same time with the velocity at B continued uniform. Hence, the

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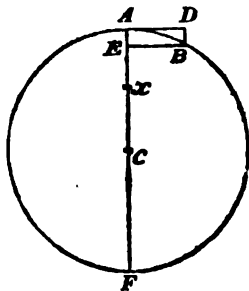
vel. of the body at A : vel. acquired thro' DB :: AB : $2DB$
and vel. thro' DB : vel. acquired thro' Ax :: $\sqrt{DB}$: $\sqrt{Ax}$;

$\therefore$ vel. at A : vel. acquired through Ax :: AB : $2\sqrt{DB \times Ax}$;

$\therefore$ by the hypothesis $AB = 2 \times \sqrt{DB \times Ax}$,

and $Ax = \dfrac{AB^2}{4DB} = \dfrac{AB^2}{4AE} = \dfrac{AF}{4}$; or a body must fall

through a space $= \dfrac{AF}{4}$, when urged by the constant force

at A, to acquire the velocity in the circle.

COR. 1.   If $m = 16\frac{1}{12}$, and r = radius, the velocity at

$A = \sqrt{\dfrac{4mr}{2}} = \sqrt{2mr} = $ the space uniformly described

in $1''$, the force at A being assumed $= 1$.

COR. 2.   If $c = $ the circumference of a circle to the
radius r, and $p = 3.14159$ &c. $=$ the circumference of a
circle to the diameter 1, $c = 2pr$; hence, to find the
whole time of revolution, we have $\sqrt{2mr}$: $2pr$ :: $1''$: T''';

$$\therefore \text{ the time } = \dfrac{2pr}{\sqrt{2mr}} = p \times \sqrt{\dfrac{2r}{m}}.$$

In the following Problems, the velocities are deter-
mined from the equation $\dot{z} = \pm F\dot{x}$. It might have been
applied in the cases preceding, but it was thought ex-
pedient to give Examples of both methods.

EXAMPLES.

Ex. 1.   To find how far a body must fall externally to
acquire the velocity in a circle, the force varying directly
as the distance from the center.

Let P be the point from which the body must fall, CB any variable distance; suppose the force at $A=1$, and $CA=a$, $CP=p$, $CB=x$. The force at $B=\dfrac{x}{a}$; $\therefore$ if z represent the space through which the body must fall by the constant force at A to acquire the proper velocity, $\dot{z} = \dfrac{-x\dot{x}}{a}$, and $z = \dfrac{-x^2}{2a}+$ corr. Let $z=0$, $V=0$;
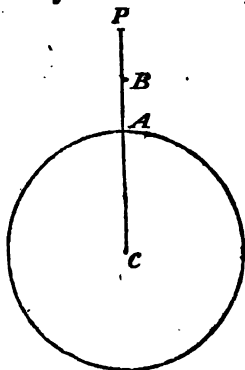
$\therefore x = cP = p$; hence, $z = \dfrac{p^2 - x^2}{2a}$; and when $x=cA$, or the body comes to A,

$z = \dfrac{p^2}{2a} - \dfrac{a}{2}$. But $z = \dfrac{a}{2}$, (by Art. 160.); $\therefore \dfrac{p^2}{2a} - \dfrac{a}{2} = \dfrac{a}{2}$;

hence, $p^2 = 2a^2$, and p or $cP = a \times \sqrt{2}$.

Ex. 2. To find how far a body must fall internally to acquire the velocity in a circle, the force varying directly as the distance from the center.

Suppose P the point to which it must descend, and let $CB = x$, a variable distance. The force at $B=\dfrac{x}{a}$; $\therefore \dot{z} = \dfrac{-x\dot{x}}{a}$, and $z = \dfrac{-x^2}{2a} +$ corr. Let $x=a$; then $v = 0$, and therefore $z = 0$. Hence, $C = \dfrac{a^2}{2a} = \dfrac{a}{2}$; $\therefore z = \dfrac{a}{2} -$
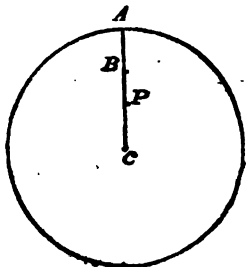
$\dfrac{x^2}{2a}$, and when $x=p$, or the body comes down to P,

$x = \dfrac{a}{2} - \dfrac{p^2}{2a}$. But $x = \dfrac{a}{2}$; $\therefore \dfrac{a}{2} = \dfrac{a}{2} - \dfrac{p^2}{2a}$, and $\dfrac{p^2}{2a} = 0$,

that is, $p = 0$; $\therefore$ the body must fall to the center.

Ex. 3. To find how far a body must fall externally to acquire the velocity in a circle, the force varying $\dfrac{1}{\text{dist.}}$ from center.

The force at $B = \dfrac{a}{x}$; $\therefore \dot{z} = \dfrac{-a\dot{x}}{x}$, and $z = -a \times$ hyp.

log. $x + $ corr. $= a \times$ h. log. $\dfrac{p}{x}$, (when the body comes to A)

$= a \times$ hyp. log. $\dfrac{p}{a}$; $\therefore \dfrac{a}{2} = a \times$ hyp. log. $\dfrac{p}{a}$; $\therefore$ hyp. log.

$\dfrac{p}{a} = \dfrac{1}{2}$. Let $e = 2.71828$ &c. = the number, whose hyperbolic logarithm is 1; then (Art. 139. Cor.) $e^{\frac{1}{2}} = \dfrac{p}{a}$;

$\therefore p = a \times e^{\frac{1}{2}} = cP.$

Ex. 4. To find the same internally.

The force at B within the circle $= \dfrac{a}{x}$; $\therefore \dot{z} = \dfrac{-a\dot{x}}{x}$, and

$z = -a \times$ hyp. log. $x + $ corr. $= a \times$ hyp. log. $\dfrac{a}{x}$, (when the

body comes to P) $= a \times$ hyp. log. $\dfrac{a}{p}$; that is, $\dfrac{a}{2} = a \times$ hyp.

log. $\dfrac{a}{p}$. Hence, assuming e as before, $p = \dfrac{a}{e^{\frac{1}{2}}} = cP.$

Ex. 5. To find how far a body must fall externally

to acquire the velocity in a circle, if the force vary as $\dfrac{1}{D^2}$ from the center.

The force at $B = \dfrac{a^2}{x^2}$; $\therefore \dot{z} = -\dfrac{a^2 \dot{x}}{x^2}$, and $z = \dfrac{a^2}{x} - \dfrac{a^2}{p}$. When the body comes to A, it $= a - \dfrac{a^2}{p}$; that is, $\dfrac{a}{2}$

$= a - \dfrac{a^2}{p}$; $\therefore \dfrac{a^2}{p} = \dfrac{a}{2}$; $\therefore p$ or $cP = 2a$.
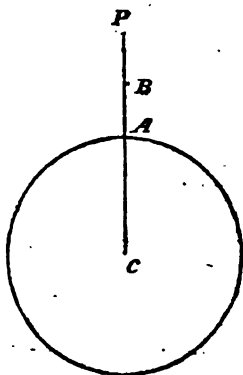
Ex. 6. To find the same internally.

The force at $B = \dfrac{a^2}{x^2}$; $\therefore \dot{z} = -\dfrac{a^2\dot{x}}{x^2}$, and $z = \dfrac{a^2}{x} - \dfrac{a^2}{a}$. Let the body come to P, and $\dfrac{a}{2} = \dfrac{a^2}{p} - a$;

$\therefore \dfrac{a^2}{p} = \dfrac{3a}{2}$, and $p = \dfrac{2a}{3}$.
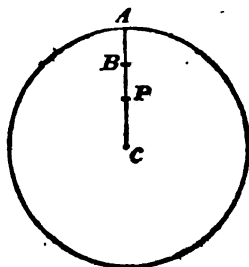
Ex. 7. To find the same externally, if the force $\propto \dfrac{1}{D^3}$.

The force at $B = \dfrac{a^3}{x^3}$; $\therefore \dot{z} = -\dfrac{a^3\dot{x}}{x^3}$, and $z = \dfrac{a^3}{2x^2} - \dfrac{a^3}{2p^2}$; or, when the body comes to P, $\dfrac{a}{2} = \dfrac{a}{2} - \dfrac{a^3}{2p^2}$; $\therefore p^2 = \dfrac{a^3}{0}$, or the body must fall from infinity to acquire the velocity at A.

Ex. 8. To find the same internally.

The force at $B = \dfrac{a^3}{x^3}$; $\therefore \dot{z} = -\dfrac{a^3\dot{x}}{x^3}$, and $z = \dfrac{a^3}{2x^2} - \dfrac{a}{2}$.

Let the body come to P, then $\dfrac{a}{2} = \dfrac{a^3}{2p^2} - \dfrac{a}{2}$; $\therefore \dfrac{a^3}{2} = p^2 \times a$,

and $p = \dfrac{a}{\sqrt{2}}$.

Ex. 9. To find how far a body must fall externally to acquire the velocity in a circle, the force varying as $\dfrac{1}{D^{n+1}}$ from the center.

The force at $B = \dfrac{a^{n+1}}{x^{n+1}}$; $\therefore \dot{z} = -\dfrac{a^{n+1}\dot{x}}{x^{n+1}}$, and $z = \dfrac{a^{n+1}}{nx^n} - \dfrac{a^{n+1}}{np^n}$; or, when the body comes to A, $\dfrac{a}{2} = \dfrac{a}{n} - \dfrac{a^{n+1}}{np^n}$; $\therefore \dfrac{a^{n+1}}{np^n} = \dfrac{a \times (2-n)}{2n}$, and $p = a \times \sqrt[n]{\dfrac{2}{2-n}}$.

Cor. 1. If $n = 1$, $F \propto \dfrac{1}{D^2}$, and $p = 2a$, as before.

Cor. 2. If $n = 2$, $F \propto \dfrac{1}{D^3}$, and $p = a \times \sqrt{\dfrac{2}{0}} =$ an infinite quantity.

Ex. 10. How far must a body fall internally, to acquire the velocity in a circle, the force varying $\dfrac{1}{D^{n+1}}$?

Here $\dot{z} = \dfrac{-a^{n+1}\dot{x}}{x^{n+1}}$, and $z = \dfrac{a^{n+1}}{nx^n} - \dfrac{a}{n}$. When the body comes to P, $\dfrac{a}{2} = \dfrac{a^{n+1}}{np^n} - \dfrac{a}{n}$, or $\dfrac{a^{n+1}}{np^n} = \dfrac{(n+2).a}{2n}$; $\therefore p = a \times \sqrt[n]{\dfrac{2}{2+n}}$.

Cor. 1. If $n = -2$, $F \propto$ dist. and $p = a \times \left.\dfrac{2}{0}\right]^{-\frac{1}{2}} = 0$, as before.

Cor. 2. If $n = 1$, $p = \dfrac{2a}{3}$, and $F \propto \dfrac{1}{D^2}$.

COR. 3. If $n=2$, $p = a \times \sqrt{\dfrac{1}{2}}$, and $F \propto \dfrac{1}{D^3}$.

If the force vary as D^{n-1}, general expressions may be deduced in the same manner.

The distance p, if the body fall externally, to acquire the velocity in a circle, $= a \times \sqrt{\dfrac{2+n}{2}}$. If it fall internally, $p = a \times \sqrt{\dfrac{2-n}{2}}$.

In all these cases, we have considered the force situated in C as attractive. If the force be repulsive, the same reasoning may be adopted to find the velocities and times of bodies ascending; only in this case, $2v\dot{v} = +4mF\dot{x}$, and $\dot{z} = +F\dot{x}$, for the velocity increases with the increase of x.

LEMMA.

(161.) The space through which a body must fall to acquire the velocity in any curve at any point P, is equal to $\frac{1}{4}$th the chord of curvature at that point; the force of acceleration being equal to the force at P.
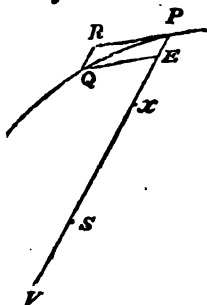
Let PV be the chord of curvature passing through S the center of force; PQ a small arc, PR a tangent at P, QR parallel to PV, and QE parallel to PR. Then, if Px be the space through which the body falls by the constant force at P to acquire the velocity at P, by the same process as in Art. 160. we have the

vel. with which PQ is described : vel. acq^d. thro' $RQ :: PQ : 2RQ$,

and vel. acquired through RQ : vel. thro' $Px :: \sqrt{RQ} : \sqrt{Px}$;

∴ vel. with which PQ is descr. : vel. thro' $Px :: PQ : 2\sqrt{RQ \times Px}$;

∴ since the first velocity = the second by hypothesis,

$$PQ = 2\sqrt{RQ \times Px}; \text{ or } Px = \frac{PR^2}{4QR} = \frac{PV}{4}.$$

COR. If the accelerating force be the same in two cases, the velocity $\propto$ the $\sqrt{}$ chord of curvature; if the force be not the same, the velocity $\propto$ as the $\sqrt{}(F \times$ the chord of curvature).

Ex. 11. How far must a body fall externally to acquire the velocity at any point P in the parabola, F varying $\frac{1}{D^2}$?

Let $c =$ the chord of curvature $= 4SP$, $SM = p$, $SB = x$, $SP = a$.

The force at $B = \frac{a^2}{x^2}$, if the force at $P = 1$; ∴ $\dot{z} = -\frac{a^2 \dot{x}}{x^2}$, and $z =$

$\frac{a^2}{x} - \frac{a^2}{p}$; ∴ when the body comes to P, $\frac{c}{4}$ or $a = a - \frac{a^2}{p}$;

∴ $\frac{a^2}{p} = 0$, or p is infinite.

Ex. 12. How far must a body fall internally, the force varying $\frac{1}{D^2}$, to acquire the velocity in a parabola ?

Here $\dot{z} = -\frac{a^2 \dot{x}}{x^2}$; ∴ $z = \frac{a^2}{x} + \text{corr.} = \frac{a^2}{x} - a$, and when

the body comes to M, $\frac{c}{4}$ or $a = \frac{a^2}{p} - a$;

∴ $2p = a$, and $p = \frac{a}{2}$.
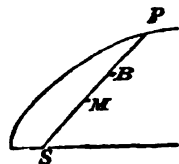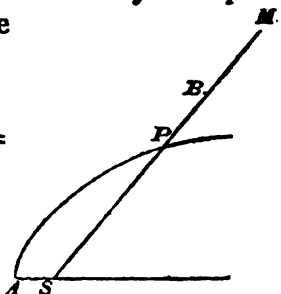
Ex. 13. How far must a body fall externally to ac-

quire the velocity in an ellipse; the force tending to the

focus, and varying $\dfrac{1}{D^4}$?

Let M be the point from which the body must fall;
assume $AN = 2a$, $SM = p$,
$SB = x$, $SP = d$, and $PH = v$.
Then, if the force at $P = 1$,

the force at $B = \dfrac{d^3}{x^3}$; $\therefore \ddot{z} =$

$-\dfrac{d^3 \dot{x}}{x^3}$, and $z = \dfrac{d^3}{x} - \dfrac{d^3}{p}$;

and when the body comes to
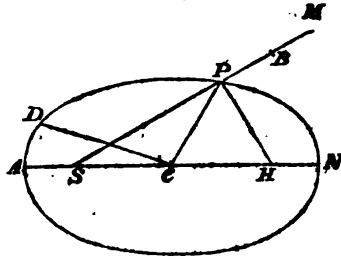
P, $\dfrac{c}{4} = d - \dfrac{d^3}{p}$. Now if CD be the semi-conjugate

diameter to the distance CP, the chord of curvature

passing through $S = \dfrac{2CD^2}{AC} = \dfrac{2SP \times HP}{AC}$ (by Conics);

$\therefore \dfrac{c}{4} = \dfrac{SP \times HP}{2AC} = \dfrac{d \times v}{2a}$; $\therefore \dfrac{d \times v}{2a} = d - \dfrac{d^3}{p}$; $\therefore \dfrac{d^3}{p} =$

$\dfrac{2ad - dv}{2a}$; or $2a \times d = p\,(2a - v) = p \times d$; $\therefore p = 2a$, or

$SM = 2AC = SP + HP$; $\therefore MP = HP$.

COR. 1. Since in whatever part of the orbit P is
assumed, the line SM always $= 2a$; if with S as a center,
and a radius $= AN$, a circle be described, a body de-
scending from any point in the circumference of that
circle, in a line MPS directly toward S, acquires a velocity
at P equal to the velocity of a body revolving in the
ellipse.

COR. 2. A body projected from P in the direction
SPM, with the velocity in the ellipse at P, will just rise

to the circumference of that circle before it loses all its velocity.

COR. 3. If BC be the semi-minor axis, and a circle be described with the center S, and radius SB, the velocity of a body revolving in that circle is equal to the velocity in the ellipse at B. For a body must fall through MB $= HB = SB$ to acquire the velocity in the ellipse at B; and (Art. 160. Ex. 5.) it falls through the same space to acquire the velocity in the circle, whose radius is SB.

EX. 14. How far must a body fall internally to acquire the velocity in the ellipse at P, under the same circumstances?

Suppose it must fall to M. Let $SM = p$, the force at $B =$

$\dfrac{d^2}{x^2}$; $\therefore \dot{z} = \dfrac{-d^2 \dot{x}}{x^2}$, and

$z = \dfrac{d^2}{x} - d.$ When the body comes to M,

$\dfrac{c}{4} = \dfrac{d^2}{p} - d,$ or $\dfrac{d \times v}{2a} =$

$\dfrac{d^2}{p} - d; \therefore d \times \dfrac{2a+v}{2a} =$

$\dfrac{d^2}{p}$, and $p = \dfrac{2ad}{2a+v} = SM;$ $\therefore PM = SP - SM = d -$

$\dfrac{2ad}{2a+v} = \dfrac{d \times v}{2a+v} = \dfrac{SP \times HP}{AN+HP}.$

Cor. Since $PM = \dfrac{SP \times HP}{AN + HP}$; $\therefore PM : HP :: SP$

$: AN + HP$. Hence, if PH be produced to L, and $HL = AN$, and HM be drawn parallel to SL, PM is that part of SP through which a body must fall to acquire the velocity at P. For $PM : HP :: SP : PL$.

Ex. 15. How far must a body fall externally to acquire the velocity in an hyperbola; the force tending to the focus, and varying $\dfrac{1}{D^2}$?

The same suppositions being made as before, the force at $B = \dfrac{d^2}{x^2}$; $\therefore z = \dfrac{d^2}{x} - \dfrac{d^2}{p}$; and

when the body comes to P, $\dfrac{c}{4} =$

$d - \dfrac{d^2}{p}$; that is, $\dfrac{d \times v}{2a} = d - \dfrac{d^2}{p}$;

$\therefore \dfrac{d^2}{p} = d - \dfrac{d \times v}{2a} = \dfrac{d(2a - v)}{2a} =$

$\dfrac{d \times -d}{2a}$; $\therefore p = -2a$. That
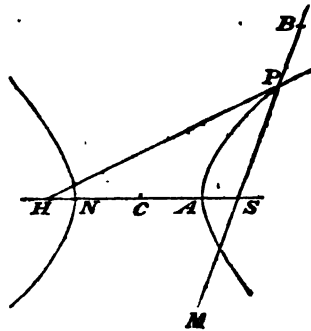
is, if PS be produced to M, and SM be taken $= 2a$, the body by ascending from M to an infinite distance, the force being considered repulsive, and then descending by the attractive force to P, will acquire the velocity at P in the curve.

Ex. 16. How far must a body fall internally to acquire the velocity in the hyperbola at P, the force varying $\dfrac{1}{D^2}$?

The force at $B = \dfrac{d^2}{x^2}$; $\therefore z = \dfrac{d^2}{x} - d$; and when the body

comes to M, $\dfrac{c}{4}$ or $\dfrac{d \times v}{2a} = \dfrac{d^2}{p} - d$; $\therefore \dfrac{d^2}{p} = \dfrac{(2a+v)d}{2a}$, or $p =$

$\dfrac{2a \times d}{2a+v}$; $\therefore PM = d - \dfrac{2a \times d}{2a+v} = \dfrac{d \times v}{2a+v} = \dfrac{HP \times SP}{AN + HP}$.

Cor. If in PH produced, HL be taken $= AN$, SL be joined, and HM drawn parallel to it, it cuts off from PS a part PM through which a body must fall to acquire the velocity in the hyperbola at P.

Proposition.

Corollary to the Three last Cases.

A body is projected at a given angle, with a given velocity, and at the distance SP from the center of force; to determine the curve in which it will move, the force in S varying $\dfrac{1}{SP^2}$.

1. Let the velocity be equal to that acquired in falling through a finite distance MP. Let Py be the direction of projection; produce yP to z; join SP. Make the angle zPH equal to yPS; take $PH = PM$; and with S, H as foci, and $SP + PH$ as major axis, describe an ellipse. That ellipse is the curve required.

2. Let the velocity equal that acquired in falling from an infinite distance to P, and Py the direction of projection. Join SP; make the angle MPy equal the angle SPy. Take $PM = SP$, and draw ML perpendicular to PM; then with S as focus, and ML directrix, describe a parabola. That parabola is the curve required.
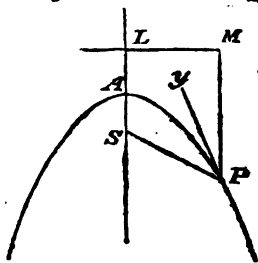
3. Let the velocity equal that acquired by ascending from some point C to infinity by a repulsive force, and then descending by the attractive force to P. Let Py be the direction of projection; make the angle $yPH =$ the angle yPC; draw CyH perpendicular to Py; then with S and H as foci, and an axis major $AB = SC$, describe the hyperbola PA, concave toward the center of force. This hyperbola is the curve required.
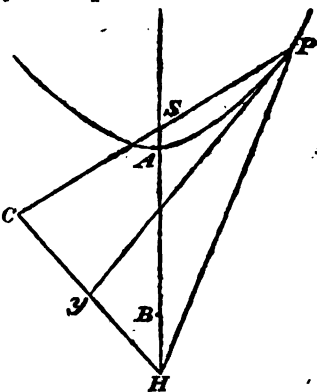
Ex. 17. How far must a body fall externally to acquire the velocity in an ellipse, the center of force being in the center of the ellipse?

Here the force varies directly as the distance from the center; $\therefore$ the force at $B = \dfrac{x}{d}$,

and $\ddot{z} = \dfrac{-x\dot{x}}{d}$; $\therefore z = \dfrac{-x^2}{2d} +$

corr. $= \dfrac{p^2}{2d} - \dfrac{x^2}{2d}$. Let the body come to P, then $\dfrac{c}{4} = \dfrac{p^2}{2d} - \dfrac{d}{2}$;

or $\dfrac{CD^2}{2d}=\dfrac{p^2}{2d}-\dfrac{d}{2}$; $\therefore CD^2=p^2-d^2$, and $p^2=CD^2+CP^2$;

$\therefore p = \sqrt{CD^2+CP^2}$.

Cor. 1. If with C as a center, and radius $=\sqrt{CD^2+CP^2}$, or $\sqrt{AC^2+BC^2}$, where BC is the semi-minor axis, a circle be described, a body descending from any point in the circumference of that circle in a line MPC, will acquire at P the velocity in the ellipse at that point,

Cor. 2. Since with the same variation of the force a body falls externally through a space $=CP \times \sqrt{2}$, to acquire the velocity in a circle of radius CP (Art. 160. Ex. 1.), the velocity in this circle and in the ellipse at P will be equal, when $\sqrt{CD^2+CP^2} = CP \times \sqrt{2}$, or when $CD = CP$.

Ex. 18. How far must a body fall internally, under the same circumstances, to acquire the velocity in the ellipse at P?

The force at $B=\dfrac{x}{d}$; $\therefore \dot{z}=$

$\dfrac{-x\dot{x}}{d}$, and $z = \dfrac{-x^2}{2d}+\dfrac{d}{2}$; $\therefore$

when the body comes to M,

$\dfrac{c}{4}$ or $\dfrac{CD^2}{2d}=\dfrac{d}{2}-\dfrac{p^2}{2d}$; hence $p^2 = CP^2 - CD^2$; $\therefore p = \sqrt{CP^2 - CD^2}$.

Cor. 1. If $CD=CP$, $p=0$. In this case, the body falls to the center; but it must fall to the center to acquire the velocity in a circle of radius CP (Art. 160. Ex. 2.); whence it appears also, as in the last Case, that if $CD=CP$, the velocity in an ellipse at P is equal to that in a circle at the same distance.

Cor. 2. If CP is less than CD, p becomes impossible; that is, the body by falling to C cannot acquire the velocity in the ellipse at P.

Ex. 19. If a body be projected obliquely, with the velocity acquired through a finite space d, by a force which varies as the distance from the center, it will describe an ellipse.

For $v\dot{v} \propto - F\dot{x} \propto - x\dot{x}$
(Art. 159. Cor. 2.); $\therefore v^2 \propto$
$d^2 - x^2$, and $v \propto \sqrt{d^2 - x^2}$.

But (Art. 158.) $v \propto \dfrac{1}{Cy}$; $\therefore$

$Cy \propto \dfrac{1}{\sqrt{d^2 - x^2}}$, which is a pro-

perty of the ellipse, where $d^2 = AC^2 + BC^2$.
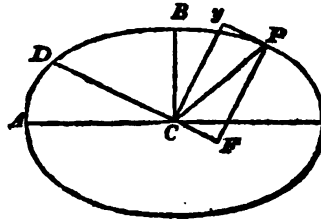
Ex. 20. A body revolves in the hyperbola AP by a repulsive force varying as the distance, of which the center is in the center of the hyperbola. To find how far it must rise from P, in the direction CPM, to acquire the velocity in the hyperbola at P.
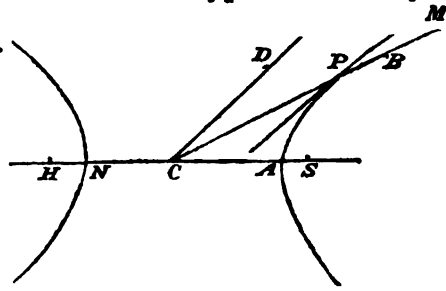
Here $\ddot{z} = + F\dot{x} = \dfrac{x\dot{x}}{d}$; $\therefore z = \dfrac{x^2}{2d} - \dfrac{d^2}{2d}$; that is, when

the body comes to M, $\dfrac{CD^2}{2d} = \dfrac{p^2 - d^2}{2d}$; $\therefore p^2 = CD^2 + d^2$, and

$p = \sqrt{CD^2 + CP^2} = CM$.

Ex. 21. The same supposition being made; to find

from what point M a body must rise to P, to acquire the velocity in the hyperbola at P.

Here $\dot{z} = \dfrac{x\dot{x}}{d}$, and $z = \dfrac{x^2}{2d}$

$- \dfrac{p^2}{2d}$. Let the body come to

P, then $z = \dfrac{d^2 - p^2}{2d}$, or $\dfrac{CD^2}{2d}$

$= \dfrac{d^2 - p^2}{2d}$; $\therefore p = \sqrt{CP^2 - CD^2} = CM$.
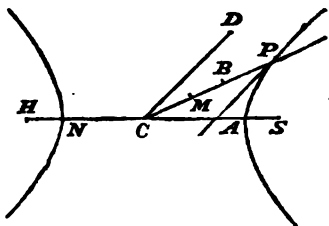
Cor. If $CP = CD$, or the hyperbola be rectangular, $CM = 0$.

Ex. 22. To determine the orbit described by a body, if projected from A in a direction perpendicular to AC, with such a velocity as would be destroyed by descending to the center C; the force being repulsive, and varying directly as the distance from C.

Suppose a body to ascend from A in the line CAG with the velocity in the curve at the point A; then, by Art. 159, the velocities in the curve and the line are equal at all other equal distances from the center of force. Now in the line, $v\dot{v} \propto F\dot{x} \propto x\dot{x}$; $\therefore v^2 \propto x^2$, and $v \propto x$, for the fluent needs no correction. Hence, if $CP = x$, the velocity in the curve at $P \propto$ as CP. But if Cy be a perpendicular

on the tangent Py, the velocity at $P \propto$ as $\dfrac{1}{Cy}$ (Art. 158.);

$\therefore CP \propto$ as $\dfrac{1}{Cy}$, which is a property of the rectangular hyperbola.
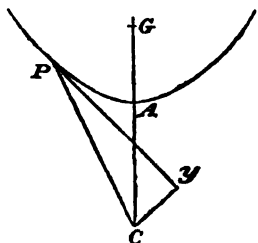
Ex. 23. Universally, the force varying as $\dfrac{1}{D^{n+1}}$; to find how far a body must fall externally, to acquire the velocity in any curve at a given point P.

Let M be the point required; take $SM = p$, SB a variable distance $= x$, $SP = d$, and $c =$ the chord of curvature at P. Then the force at P being assumed $= 1$, $\dot{z} = \dfrac{-d^{n+1}\dot{x}}{x^{n+1}}$; $\therefore$

$z = \dfrac{d^{n+1}}{nx^n} - \dfrac{d^{n+1}}{np^n}$; or, when the body comes to

$P, \dfrac{c}{4} = \dfrac{d}{n} - \dfrac{d^{n+1}}{np^n}$; $\therefore \dfrac{d^{n+1}}{np^n} = \dfrac{d - \frac{1}{4}nc}{n}$, and

$p = \dfrac{d \times d^{\frac{1}{n}}}{\sqrt[n]{d - \frac{1}{4}nc}} = SM$.
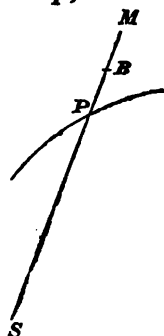
EXAMPLES.

1. Suppose the curve a circle, and $F \propto$ distance; $c = 2d$, and $n = -2$; $\therefore SM = d \times d^{-\frac{1}{2}} \times \sqrt{d + d} = \dfrac{d \times \sqrt{2d}}{\sqrt{d}}$

$= d \times \sqrt{2}$.

2. Let $F \propto \dfrac{1}{D^2}$, the curve being a circle; here $n + 1$

$= 2$; $\therefore n = 1$, and $SM = \dfrac{d \times d}{d - \dfrac{d}{2}} = 2d$.

3. Let $F \propto \dfrac{1}{D^3}$, $n + 1 = 3$; $\therefore n = 2$, and $SM = \dfrac{d \times d^{\frac{1}{2}}}{\sqrt{d - d}}$

$=$ an infinite quantity.

4. If the curve be a logarithmic spiral, $c = 2d$; and

the $F \propto \dfrac{1}{D^3}$; $\therefore SM = \dfrac{d \times d^{\frac{1}{2}}}{\sqrt{d-d}} =$ an infinite quantity.

5. In a parabola, $c = 4d$; and $F \propto \dfrac{1}{D^2}$, or $n = 1$;

$\therefore SM = \dfrac{d \times d}{d-d} =$ an infinite quantity.

6. If the curve be a circle, and the center of force in the circumference, or $F \propto \dfrac{1}{D^5}$, $n+1 = 5$; $\therefore n = 4$, and

$c = d$; hence, $SM = \dfrac{d \times d^{\frac{1}{4}}}{\sqrt[4]{d-d}} =$ an infinite quantity.

Ex. 24. The force varying as $\dfrac{1}{D^{n+1}}$; to find how far a body must fall internally, to acquire the velocity in any curve at a given point B.

Here $\dot{z} = \dfrac{-d^{n+1}\dot{x}}{x^{n+1}}$; $\therefore z = \dfrac{d^{n+1}}{nx^n} - \dfrac{d}{n}$; or, when the body

comes to M, $\dfrac{c}{4} = \dfrac{d^{n+1}}{np^n} - \dfrac{d}{n}$; and $\dfrac{d^{n+1}}{np^n} = \dfrac{d + \frac{1}{4}nc}{n}$;

$\therefore p = \dfrac{d \times d^{\frac{1}{n}}}{\sqrt[n]{d + \frac{1}{4}nc}} = SM.$
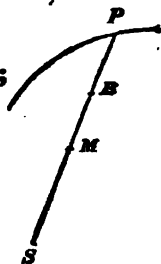
EXAMPLES.

Ex. 1. Suppose the curve a circle, and that the force varies as the distance; $c = 2d$, and $n+1 = -1$; $\therefore n = -2$; and $p = d \times d^{-\frac{1}{2}} \times \sqrt{d-d} = 0$, or the body falls to the center.

2. Let $F \propto \dfrac{1}{D^2}$, $n+1=2$; $\therefore n=1$, and $p = \dfrac{d^2}{d + \dfrac{d}{2}} = \dfrac{2d}{3}$.

3. Let $F \propto \dfrac{1}{D^3}$, $n=2$; $\therefore p = \dfrac{d \times \sqrt{d}}{\sqrt{d+d}} = \dfrac{d}{\sqrt{2}}$.

4. In the logarithmic spiral, $c = 2d$, and $F \propto \dfrac{1}{D^3}$;

$\therefore n=2$, and $p = \dfrac{d \times \sqrt{d}}{\sqrt{d+d}} = \dfrac{d}{\sqrt{2}}$.

5. In the parabola, F varies as $\dfrac{1}{D^2}$; $\therefore n=1$, and $c=4d$;

$\therefore p = \dfrac{d^2}{d+d} = \dfrac{d}{2}$.

6. In a circle, which has the center of force in the circumference, $n+1=5$; $\therefore n=4$, and $c=d$; hence,

$p = \dfrac{d \times d^{\frac{1}{2}}}{\sqrt[4]{d+d}} = \dfrac{d}{2|^{\frac{1}{4}}}$.

SECTION IV.

PROPOSITION.

(162.) To compare the velocity in a curve at any point, with the velocity in a circle, at the same distance from the center of force.

The velocity in the curve at P, is equal to that which a body would acquire, in falling through one-fourth of the chord of curvature at that point, if urged by the constant force at P; and the velocity in the circle at P is equal to that acquired through one-fourth of its diameter by the same force; hence, $V^2 \propto$ the chords of cur-

vature; or V^2 in the curve at P : V^2 in the circle SP
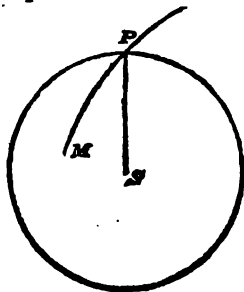
$$:: \frac{2p\dot{x}}{\dot{p}} : 2x$$

(Art. 119.) $$:: \frac{\dot{x}}{x} : \frac{\dot{p}}{p}.$$

EXAMPLES.

Ex. 1. To compare the velocity in a logarithmic spiral with the velocity in a circle at the same distance SP.

Here $x : p :: a : b$; $\therefore x : p :: \dot{x} : \dot{p}$, and $\frac{\dot{x}}{x} = \frac{\dot{p}}{p}$; hence the velocity in the spiral is equal to that in a circle at the same distance.

Ex. 2. To compare the same in a parabola.

In this case, $p^2 = ax$; $\therefore 2p\dot{p} = a\dot{x}$; also $x = \frac{p^2}{a}$; $\therefore \frac{\dot{x}}{x} =$

$\frac{2ap\dot{p}}{ap^2} = \frac{2\dot{p}}{p}$; $\therefore V^2$ in the parabola : V^2 in the circle $:: \frac{2\dot{p}}{p}$

$: \frac{\dot{p}}{p} :: 2 : 1$; and V in the parabola : V in the circle at

the same distance $:: \sqrt{2} : 1$.

Ex. 3. To compare the same in an ellipse round the focus.

Here $Sy^2 = BC^2 \times \frac{SP}{HP}$; or, if $AC = a, BC = b, p^2 = \frac{b^2 x}{2a - x}$;

$\therefore 2p\dot{p} = \frac{2ab^2\dot{x}}{(2a - x)^2} = \frac{b^2 x}{2a - x} \times$

$\frac{2a\dot{x}}{x(2a - x)} = \frac{p^2 \times 2a\dot{x}}{x(2a - x)}$. Hence,

$\frac{\dot{x}}{x} = \frac{\dot{p}}{p} \times \frac{2a - x}{a}$;
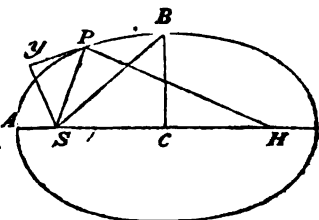
$\therefore V^2$ in the ellipse at $P : V^2$ in the circle $SP :: \dfrac{\dot{p}}{p} \times \dfrac{2a-x}{a} : \dfrac{\dot{p}}{p}$;

$$:: HP : AC ;$$

and V in the ellipse at $P : V$ in the circle $SP :: \sqrt{HP} : \sqrt{AC}.$

Cor. The velocity at B in the ellipse is equal to the velocity in a circle whose radius is SB.

In this and some other cases, it is more convenient to determine the ratio of the velocities by comparing the chords of curvature. For V^2 varies as the chord of curvature, when the accelerating force is the same. (Art. 161. Cor.)

In the two following Propositions, the velocities are compared by determining their actual values.

Ex. 4. To compare the velocity acquired by a body in falling from a given point M through a finite distance MP, with the velocity of a body in a circle at the distance SP, the force varying as $\dfrac{1}{D^{n+1}}$.
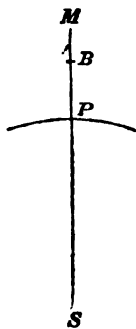
Let $SP = d$, and the force at $P = 1$; $SM = p$, SB a variable distance $= x$. Then the force at $B = \dfrac{d^{n+1}}{x^{n+1}}$;

$\therefore \dot{z} = -\dfrac{d^{n+1}\dot{x}}{x^{n+1}}$, and $z = \dfrac{d^{n+1}}{nx^n} - \dfrac{d^{n+1}}{np^n} = \dfrac{d^{n+1} \times (p^n - x^n)}{np^n x^n}.$

But $V^2 = 4mz$; $\therefore V^2$ at P, where $x = d, = 4md \times \dfrac{p^n - d^n}{np^n}.$

Also V^2 in a circle at the distance $SP = 4m \times \dfrac{d}{2}$;

$\therefore$ velocity of body falling at P : velocity in circle SP

$$:: \sqrt{\frac{4md \times \overline{p^n - d^n}}{np^n}} : \sqrt{\frac{4md}{2}}$$

$$:: \sqrt{p^n - d^n} : \sqrt{\frac{np^n}{2}}.$$

Cor. 1. When the velocity of the falling body is equal to the velocity in a circle at that distance, $p^n - d^n = \dfrac{np^n}{2}$, and p may be found; thus, if the force $\propto \dfrac{1}{D^2}$, since $n = 1$, $p = 2d$.

Cor. 2. If $F \propto \dfrac{1}{D^2}$,

vel. of body falling in P : vel. in circle SP :: $\sqrt{p - d} : \sqrt{\dfrac{p}{2}}$

$$:: \sqrt{MP} : \sqrt{\tfrac{1}{2}MS}.$$

(Newton, Sect. 7. Prop. 33.)

Ex. 5. The force varying according to the same law; to compare the velocity of a body falling from an infinite distance to a given point P, with the velocity in a circle at the distance SP.

By proceeding as in the last case, $z = \dfrac{d^{n+1}}{nx^n}$, and the correction $= 0$; $\therefore$ when the body comes down to P, $z = \dfrac{d}{n}$; hence, velocity of the falling body at P : vel. in circle SP

$$:: \sqrt{\frac{4md}{n}} : \sqrt{\frac{4md}{2}}$$

$$:: \sqrt{2} : \sqrt{n}.$$

Examples.

Ex. 1. Let $F \propto \dfrac{1}{D^2}$, $n = 1$; therefore the

vel. from infinity : vel. in a circle SP :: $\sqrt{2}$: 1, as in the parabola.

Ex. 2 Let $F \propto \dfrac{1}{D^3}$, $n=2$; therefore the velocity from infinity is equal to the velocity in a circle SP, as in the logarithmic spiral.

Cor. 1. If the ratio of the velocity acquired from infinity at P to that in a circle be given :: c : 1, the law of the force may be found. For c : 1 :: $\sqrt{2}$: $\sqrt{n}$; $\therefore n+1 = \dfrac{2+c^2}{c^2}$.

Cor. 2. If a body be projected from P in the direction PM, with a velocity which is to the velocity in a circle at P :: $\sqrt{2}$: $\sqrt{n}$, it will go off *ad infinitum*; and this is the least velocity with which it can be projected, so that it may never return.

Cor. 3. Since $V^2 \propto$ as the chord of curvature when the force is the same, we can compare the chord of curvature of the curve described by a body projected perpendicularly at P; its velocity being that acquired from infinity, with the given distance SP.

For 2 : n :: the chord of curvature required : $2SP$; therefore the chord of curvature $= \dfrac{4SP}{n}$.

Thus, if $n+1 = 2$, $n=1$, and the chord $= 4SP$; or the curve is a parabola.

Cor. 4. If the force vary inversely as the cube of the distance, $n=2$; and a body projected at P, in a direction perpendicular to SP with the velocity acquired from infinity, will in this case describe a circle. If the force vary in a higher inverse ratio than $\dfrac{1}{D^3}$, 2 is less than n;

hence, the velocity of the body is less than that in a circle at the same distance; but $V^2 \propto PV$, where the force is given. Hence, the chord of curvature of the curve described is less than that of a circle, whose radius is SP; and the body will describe an orbit interior to the circle, and at last fall into the center. If the force vary in a less inverse ratio than $\dfrac{1}{D^3}$, 2 is greater than n, or the velocity of the body is greater than that in a circle of radius SP; therefore, since $V^2 \propto PV$, the chord of curvature of the curve described is greater than the diameter of the circle; hence, the body will describe an orbit exterior to the circle, and at length go off to infinity. The two following Problems are connected with this subject.

PROB. 1.

(163.) If the force vary as $\dfrac{1}{D^n}$ from the center, where n is greater than 3, and a body be projected from V in a direction perpendicular to the line SV, with the velocity acquired in falling from an infinite distance, it is required to find after how many revolutions it will fall into the center.

Let VPp be the curve described; VRr a circle, whose radius is SV; SPR, SPr two lines drawn from S indefinitely near to each other. Draw the tangent Py, Sy perpendicular to it, and px perpendicular to SP. Take $SP = x$, $SV = r$. Then in the curve (Art. 159. Cor. 2.) $v\dot{v} \propto -F\dot{x}$

$$\propto \frac{-\dot{x}}{x^n}; \quad \therefore V^2 \propto \frac{1}{x^{n-1}},$$
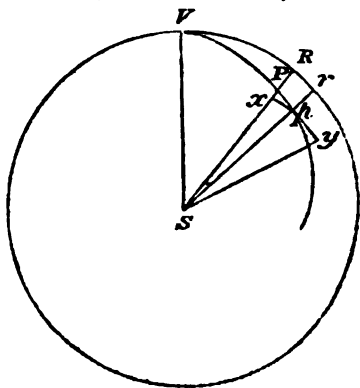
and the correction vanishes. But, by Art. 158. $V \propto$ inversely as the perpendicular.

Hence, V^2 at $V : V^2$ at $P :: x^{n-1} : r^{n-1} :: Sy^2 : SV^2$;

$$\therefore Sy^2 = \frac{r^2 x^{n-1}}{r^{n-1}} = \frac{x^{n-1}}{r^{n-3}}; \quad \therefore Py^2 = SP^2 - Sy^2 = x^2 - \frac{x^{n-1}}{r^{n-3}} =$$

$$\frac{x^2 r^{n-3} - x^{n-1}}{r^{n-3}}. \quad \text{Now, by similar triangles, } Ppx, \ SPy,$$

$$Px : px :: Py : Sy;$$
$$\text{and } px : Rr :: SP : SR;$$
$$\therefore Px : Rr :: SP \times Py : Sy \times SR; \text{ or, if } VR = z,$$
$$-\dot{x} : \dot{z} :: \sqrt{x^4 r^{n-3} - x^{n+1}} : \sqrt{r^2 x^{n-1}};$$

$$\therefore \dot{z} = \frac{-r x^{\frac{n-1}{2}} \dot{x}}{x^2 \sqrt{r^{n-3} - x^{n-3}}} = \frac{-r x^{\frac{n-5}{2}} \dot{x}}{\sqrt{r^{n-3} - x^{n-3}}}; \text{ whose fluent}$$

$(f. 48.)$ or $z = \dfrac{2r}{(n-3) r^{\frac{n-3}{2}}} \times$ a circular arc of radius

$= r^{\frac{n-3}{2}}$, and cosine $x^{\frac{n-3}{2}}$, which needs no correction.

Let $x = 0$, and $r = 1$; then $z = \dfrac{2}{n-3} \times$ a quadrant $= \dfrac{2}{n-3}$

$\times \dfrac{\text{circumference}}{4} = \dfrac{\text{circumference}}{2n-6}$; that is, the body

describes $\dfrac{1}{2n-6}$ revolutions, before it falls into the center.

Cor. If $n = 4$, the body falls into the center after half a revolution.

PROB. 2.

(164.) If the force $\propto \dfrac{1}{D}$ from the center, where n is greater than 1 and less than 3, and a body be projected as in the

last case; to find after how many revolutions it will go off *ad infinitum.*

In this case, $\dot{x}$ is positive; and by reasoning in the same manner as before, Rr or $\dot{z}$ is found $=$ $\dfrac{r x^{\frac{n-5}{2}} \dot{x}}{\sqrt{r^{n-3} - x^{n-3}}}$; and the fluent $(f.\ 48.) =$ $-\dfrac{\times\ 2r}{(n-3)\ r^{\frac{n-3}{2}}} \times$ circular arc of radius $= r^{\frac{n-3}{4}}$, and cosine $x^{\frac{n-3}{2}}$, which

needs no correction. When r is unity, and x is infinite (or when the cosine $= 0$, since n is less than 3, and therefore $\dfrac{n-3}{2}$ negative), it $= \dfrac{2}{3-n} \times$ a quadrant $= \dfrac{1}{6-2n}$ revolutions.

Cor. If $n = 1$, $\dfrac{1}{6-2n} = \dfrac{1}{4}$, or the body goes off *ad inf.* after $\dfrac{1}{4}$ of a revolution. This must be considered as the limit, or the least angle at which the body can go off *ad infinitum;* for n cannot be assumed accurately $= 1$, since in this case the $f.\dfrac{-\dot{x}}{x^n}$ cannot be taken by the common method.

SECTION V.

PROPOSITION.

(165.) If a body describe a curve, when urged by a

force acting in the direction of the ordinates, that force is, proportional to the second fluxion of the ordinate.

Let PR be a tangent to the curve at P; and MP, NQR, two ordinates indefinitely near to each other, in the direction of the force. The body at P would describe PR, if the force did not act, in the same time that it now describes PQ; therefore RQ represents the small space, through which it is impelled by the force, in the time of describing PQ. Now, ultimately, when the time is very small, $S \propto F \times T^2$; therefore, if T be assumed a small given time, $RQ \propto F$; but ultimately $RQ = \pm \frac{1}{2}\ddot{y}$, if $MP = y$ (Art. 97.); $\therefore F \propto \pm \ddot{y}$, according as the force is repulsive or attractive.
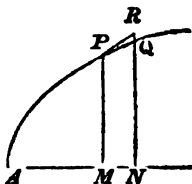
EXAMPLES.

Ex. 1. A body describes a curve, whose equation is $x^5 = a^3 y^2$, by a force acting in the direction of the ordinates. Required the law.

Here $y \propto x^{\frac{5}{2}}$; $\therefore \dot{y} \propto x^{\frac{3}{2}}\dot{x}, \propto x^{\frac{3}{2}}$, since the motion in a direction parallel to the axis must in these cases be uniform. Therefore $\ddot{y} \propto x^{\frac{1}{2}}$; that is, the force $\propto x^{\frac{1}{2}}$, or $\propto y^{\frac{1}{5}}$, and is repulsive; or the curve is convex to its axis.

Ex. 2. Let the equation to the curve be $y^{m+n} = a^m x^n$, and act in the direction of the ordinates. To find the law.

Here $y \propto x^{\frac{n}{m+n}}$; $\therefore \dot{y} \propto x^{\frac{-m}{m+n}}$, and $\ddot{y} \propto -x^{-\frac{2m+n}{m+n}}$,

$\propto -\dfrac{1}{x^{\frac{2m+n}{m+n}}} \propto -\dfrac{1}{y^{\frac{2m+n}{n}}}$, and is attractive; or the curve is concave to the axis.

Ex. 3. A body describes a semi-circle by the action

of a force in the direction of the ordinates. Required the law.

If $AM = x$, $AC = a$, $PM = y$,

$$y = \sqrt{2ax - x^2};$$

$$\therefore \dot{y} = (a - x)\, \dot{x}\, (2ax - x^2)^{-\frac{1}{2}}$$

$$= (a - x) \cdot (2ax - x^2)^{-\frac{1}{2}}$$
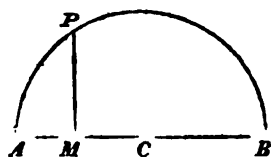
and $\ddot{y} = -\dot{x}(2ax - x^2)^{-\frac{1}{2}} - \dfrac{(a-x)^2}{(2ax - x)^{\frac{3}{2}}} \times \dot{x} =$

$-\dfrac{(2ax - x^2) + (a-x)^2}{(2ax - x^2)^{\frac{3}{2}}} = \dfrac{-a^2}{y^3}$; therefore the force $\propto \dfrac{1}{PM^3}$.

The same conclusion is deduced from the expression $\dfrac{\dot{p}}{p^3 \dot{x}}$. Art. 153. Ex. 8.

Ex. 4. Let the curve be a parabola, and the force act as before.

In the parabola, $y = a^{\frac{1}{2}} x^{\frac{1}{2}}, \propto x^{\frac{1}{2}}$; $\therefore \dot{y} \propto x^{-\frac{1}{2}}$; $\therefore -\ddot{y} \propto x^{-\frac{3}{2}}$ $\propto \dfrac{1}{y^3}$; or the force $\propto \dfrac{1}{y^3}$.

Ex. 5. Let the curve be an ellipse.

Here $y = \dfrac{b}{a} \times (2ax - x^2)^{\frac{1}{2}} \propto (2ax - x^2)^{\frac{1}{2}}$; hence, the reasoning and the conclusion is the same as in the circle, and the force $\propto \dfrac{1}{y^3}$.

Ex. 6. Let the curve be an hyperbola.

Here $y = \dfrac{b}{a} \times (2ax + x^2)^{\frac{1}{2}} \propto (2ax + x^2)^{\frac{1}{2}}$; and $\dot{y} \propto (a+x)$

$(2ax + x^2)^{-\frac{1}{2}}$; $\therefore \ddot{y} \propto \dfrac{2ax + x^2 - \overline{a + x}|^2}{(2ax + x^2)^{\frac{3}{2}}}$; and $-\ddot{y} \propto$

$\dfrac{1}{(2ax + x^2)^{\frac{3}{2}}} \propto \dfrac{1}{y^3}$.

Ex. 7. Let the curve be a common cycloid VDC, and the force act in the direction of the ordinates, parallel to the axis.

Let DE, and PL, be two ordinates indefinitely near to each other; draw DM parallel to CA, meeting the axis in M, and the generating circle in R; join VR, AR, and let $CE = x$, $ED = y$, $Dn = \dot{x}$, $Pn = \dot{y}$; and $AV = 2a$.
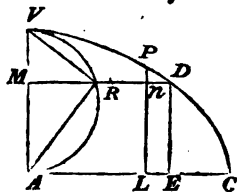
Then, by similar triangles, PDn, RVM, $\dot{x} : \dot{y} :: RM :$

$$MV :: AM : MR :: y : \sqrt{2ay - y^2}; \therefore \dot{y} = \dot{x} \times \frac{(2ay - y^2)^{\frac{1}{2}}}{y};$$

$$\therefore \ddot{y} = \frac{(a\dot{y} - y\dot{y})\, y - \dot{y}\, (2ay - y^2)}{y^2 (2ay - y^2)^{\frac{1}{2}}} = \frac{-ay\dot{y}}{y^2 (2ay - y^2)^{\frac{1}{2}}}.$$

But $\dot{x} = \frac{y\dot{y}}{(2ay - y^2)^{\frac{1}{2}}}$; $\therefore \ddot{y} = \frac{-a\dot{x}}{y^4}$; and $-\ddot{y} \propto \frac{1}{y^4}$, or the

force $\propto \dfrac{1}{y^4}$.

SECTION VI.

PROPOSITION.

(166.) To find the forces, by which bodies may be made to perform isochronous oscillations in curve lines, when the force tends to a center.

Let AT represent a pendulum, C the center of force, and R the lowest point in the curve. By Art. 157. Ex. 8. Cor. 3., if the force acting upon T in a tangential direction be proportional to TR, the oscillations will be isochronous. Join CT; draw Ty a tangent to the curve; take $Ty = TR$, and from y let yz be drawn perpendicular to Ty. Then, if Tz represent the force tending to C, the oscillations will be isochronous.
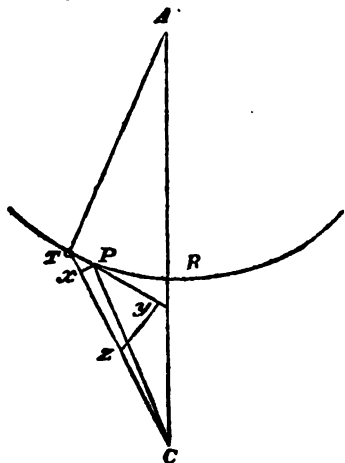
Take CP very near CT, and draw Px perpendicular to CT. Let $RT=z$, $CT=x$, $Tz=F$, $TP=\dot{x}$, $Tx=\dot{z}$. Then, by similar triangles, F : the tangential force Ty :: $\dot{z} : \dot{x}$; $\therefore F = \dfrac{\text{the tangential force} \times \dot{z}}{\dot{x}} \propto \dfrac{z\dot{z}}{\dot{x}}$.

EXAMPLES.

Ex. 1. Let the force vary as the distance; then $z\dot{z}$ varies as $x\dot{x}$: $\therefore z^2$ varies as x^2, and z varies as x, or the curve is the logarithmic spiral.

Ex. 2. Required the law of the force tending to S, which will make the involute of a circle the isochronous curve.

Here $\dot{z} : \dot{x} :: ST : Ty :: x : r$;

$\therefore \dfrac{\dot{z}}{\dot{x}} = \dfrac{x}{r}$, and $\dfrac{z\dot{z}}{\dot{x}} = \dfrac{zx}{r} = F$; $\therefore F \times r$

$= z \times x$; or, F required : the arc :: the distance ST : the radius of the circle.
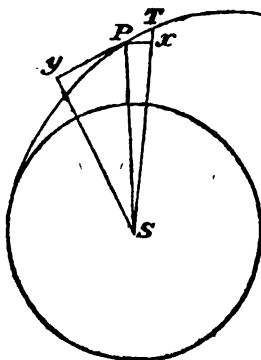
PROPOSITION.

(167.) To find an expression for the force necessary to make any curve isochronous, when it acts in parallel lines.

Let it act in lines parallel to AR. Draw Py a tangent equal to PR, yZ perpendicular to Py, and let it meet PZ drawn in the direction of the force in Z. Then, making the same assumption as before,
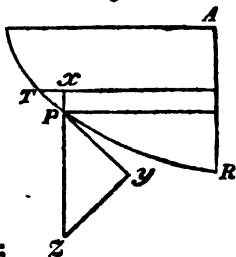
F : the tangential force :: $PZ : Py$

$:: PT : Px :: \dot{z} : \dot{x}$;

$\therefore F = \dfrac{\text{the tangential force} \times \dot{z}}{\dot{x}} \varpropto \dfrac{z\dot{z}}{\dot{x}}$.

EXAMPLES.

Ex. 1. Let the force be constant. Then $\dfrac{z\dot{z}}{\dot{x}}$ varies as 1, and x varies as z^2, a property of the common cycloid.

Ex. 2. Let TR be a circular arc; to find the law.

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The force required : the tangential force :: $\dot{s}$: $\dot{x}$ :: r : sine; or, if the tangential force be represented by the arc, and gravity by the radius of the circle, F : gravity :: arc : sine.

Ex. 3. Let the curve be the catenary, and the force act in a direction parallel to the axis.

In this curve, $z^2 = 2ax + x^2$; $\therefore z\dot{z} = a\dot{x} + x\dot{x}$. Hence F, or $\dfrac{z\dot{z}}{\dot{x}} = a + x$; therefore this force : gravity :: $a + x : a$.

CHAP. XX.

ON THE MOTION OF BODIES IN RESISTING MEDIUMS.

(168.) $\mathbf{I}$ N estimating the effects of fluids upon the motion of bodies, the retardation may be conceived to vary according to any law of the velocity. Thus, in the second Book of the *Principia*, Sect. 1, it is supposed to be proportional to the velocity itself; and any other power might be assumed at pleasure. But it appears by experiment, that the resistance opposed to a given plane surface varies nearly as V^2; so that every other law of variation is to be considered rather as a mathematical hypothesis, than as founded upon fact. We are not sufficiently acquainted with the nature of fluids, to ascertain precisely in what manner they act. Hence, in demonstrating the theory, Sir I. NEWTON has introduced the following conditions. (Lib. II. Prop. 40., &c.)

1. That the particles of the fluid, in which the body moves, are perfectly non-elastic, and

2. That the fluid is infinitely compressed.

The first property belongs to mercury and water, and, provided the bodies move slowly, even to air; for the particles may then be considered as sliding away after impact, without adhesion to the body. If the particles were elastic, an additional resistance would be produced by the rebound.

The second supposition is not strictly true of any fluid. If a body move in a medium with considerable velocity, the parts left by it, as it advances, will not immediately

be filled; and an additional resistance will arise from this cause. Thus, if its velocity in air be greater than that with which air rushes into an exhausted receiver, a vacuum will be left behind. Where the velocity is considerably greater, the air is condensed before the body, and, in proportion to the compression, exerts a force of elasticity against it. But in experiments where the velocity is slow, the second condition may be admitted without sensible error.

It is therefore assumed, that no resistance is opposed to the moving body, except that which arises from the inertia of the particles displaced; the effects of elasticity, tenacity, and friction, if any friction exist, in philosophical experiments on air, water, and mercury, being scarcely perceptible.

(169.) It is found by experiment, that the resistance opposed to a plane surface moving in a fluid, in a direction perpendicular to the plane, is equal to the weight of a column of the fluid, whose base is the area of the plane, and height the space through which a body must fall by gravity to acquire its velocity. Therefore, if $A =$ the area of the plane surface, and $z =$ the height from which a body must fall to acquire its velocity, the resistance $= A \times z$; the density of the medium being assumed $= 1$. The same reasoning is true for a cylinder moving in a fluid, in the direction of its axis; the curved surface, which is in the direction of its motion, being supposed to have no tendency to accelerate or retard the particles of the fluid. Hence, if $d =$ the diameter of its base, the resistance opposed to it $= \dfrac{z p d^2}{4}$.

(170.) Now in fluids indefinitely compressed, the resistance on this cylinder : that on a globe of the same

diameter :: 2 : 1; therefore the resistance on a globe moving with the same velocity $= \dfrac{\pi p d^2}{8}$.

(171.) The same distinction is to be noticed between resisting and retarding forces, which exists between moving and accelerating forces. The resistance opposed to a body moving in a fluid is proportional to the quantity of motion destroyed in a given time; whereas, the retarding force is measured by the velocity destroyed in that time, or it is proportional to the resistance divided by the quantity of matter.

For the sake of perspicuity, this distinction is observed in the two following Sections.

(172.) Hence, if $n =$ the density of a globe, whose diameter is d, or $\dfrac{p d^3 n}{6}$ its quantity of matter, and the specific gravity of the fluid be 1, the retarding force $=$ $\dfrac{\pi p d^2}{8} \div \dfrac{p d^3 n}{6} = \dfrac{3\pi}{4nd}$; or, in terms of the velocity, since

$V^2 = 4m\pi$, the retarding force $= \dfrac{3V^2}{16mnd}$.

It was thought expedient to give examples of both methods of calculation in the following Propositions.

SECTION I.

PROPOSITION.

(173.) The force of resistance, which is opposed to a sphere, moving in a fluid with any given velocity, is to the force which would destroy the sphere's whole motion, in the same time, in which it describes uniformly $\dfrac{8}{3}$ parts

of its diameter, as the density of the fluid to the density of the sphere.   (Newton, Lib. II. Prop. 38.)

Let M represent the resisting or moving force, which would destroy the sphere's motion in the specified time. Let Q denote the quantity of matter in the sphere, V its velocity, and d the diameter ; and suppose the specific gravity of the sphere is to that of the fluid as n to 1. Then, since a body will describe $\dfrac{8d}{3}$ by an uniform velocity V, in the same time that it describes $\dfrac{4d}{3}$ when retarded by a constant force $\dfrac{M}{Q}$, which destroys that velocity, we have $V^2 = 4m \times \dfrac{M}{Q} \times \dfrac{4d}{3}$.   But $V^2 = 4mz$, according to the notation of the preceding Chapter ; therefore $4m \times \dfrac{M}{Q} \times \dfrac{4d}{3} = 4mz$, and $M = \dfrac{3Qz}{4d}$.   Now $Q = \dfrac{pd^3 n}{6}$; therefore $M = \dfrac{zpd^2 n}{8}$.   But the resisting force opposed to the sphere $= \dfrac{zpd^2}{8}$ (Art. 170.) ; and this is to $\dfrac{zpd^2 n}{8}$ :: 1 : n, or as the specific gravity of the fluid to that of the sphere.

PROPOSITION.

(174.)   Let a sphere of given diameter be projected in a resisting medium, whose specific gravity is to that of the sphere as 1 : n.   Having given the velocity of projection at the point A, to find the velocity of the sphere at any given point C.

Let $AC=x$, $CD=\dot{x}$; then, since the retarding force $=$ $\frac{3z}{4nd}$ (Art. 172.), $\dot{z}=-\frac{3z\dot{x}}{4nd}$; $\therefore \frac{\dot{z}}{z}=-\frac{3\dot{x}}{4nd}$, and the

hyp. log. $z+$ corr. $=-\frac{3x}{4nd}+$ corr. When $x=0$, let

$z=a$; then the hyp. log. $\frac{z}{a}=-\frac{3x}{4nd}$. Hence, if $e=$

2. 71828 &c., the number, whose hyperbolic logarithm is 1, we have $\frac{z}{a}=e^{\frac{-3z}{4nd}}$; $\therefore z=a\times e^{\frac{-3z}{4nd}}$; and the ve-

locity at $C=\sqrt{4mz}=\dfrac{\sqrt{4ma}}{e^{\frac{3z}{8nd}}}$.

Cor. If the spaces be taken in arithmetic progression, the velocities are in inverse geometric.

<div style="text-align:center">PROPOSITION.</div>

(175.) To find the time of describing AC.

Here $\dot{T}$, which $=\dfrac{\dot{x}}{V}=\dfrac{\dot{x}\times e^{\frac{3z}{8nd}}}{\sqrt{4ma}}$, and $T=\dfrac{8nd}{3.\sqrt{4ma}}.e^{\frac{3z}{8nd}}$ $+$ corr.

Let $T=0$, then $x=0$; $\therefore e^{\frac{3z}{8nd}}=e^{0}=1$. Hence, $T=$ $\dfrac{8nd}{3\times\sqrt{4ma}}\times\left(e^{\frac{3z}{8nd}}-1\right)$.

<div style="text-align:center">PROPOSITION.</div>

(176.) If the retarding force, opposed to a sphere projected with a velocity c in a resisting medium, vary as the velocity, the diminution of velocity is proportional to the space described. (NEWTON, Lib. II. Prop. 1.)

Let $r\left(=\dfrac{3c^{2}}{16mnd}\right.$, Art. 172.$\left.\right)$ represent the retarding

force corresponding to the velocity c, and v be any variable velocity; then, since $c : v :: r : \dfrac{rv}{c}$; $\therefore \dfrac{rv}{c}$ is the retarding force corresponding to the velocity v. Hence,

$$v\dot{v} = -2mF\dot{x} = -\frac{2mr}{c}\times v\dot{x}; \therefore \dot{x} = -\frac{c\dot{v}}{2mr}, \text{ and } x + \text{corr.}$$

$\propto$ corr. $- v$; but if $x = 0$, $v = c$. Hence, $x \propto c - v$; or the space described varies as the velocity lost.

PROPOSITION.

(177.) If the retarding force vary as in the last case, and times be taken in arithmetic progression, the velocities at the beginning of those times are in geometric progression. (NEWTON, Lib. II. Prop. 2.)

The same assumption being made, $\dot{x} = \dfrac{-c\dot{v}}{2mr}$; and $\dot{T} =$

$$\frac{\dot{x}}{V} = \frac{-c}{2mr}\times\frac{\dot{v}}{v}; \therefore T = -\frac{c}{2mr} \times \text{ hyp. log. } v + \text{corr.}\quad \text{Let}$$

$T = 0$; then $v = c$. Hence, $T = \dfrac{c}{2mr} \times \text{ hyp. log. } \dfrac{c}{v}$; or $T \propto$ hyp. log. $\dfrac{c}{v}$.

If, therefore, the times be taken in arithmetic progression, the logarithms of the velocities are in arithmetic progression, and $\dfrac{c}{v}$, or v itself, in geometric progression.

COR. The spaces AB, BC, CD, &c. described in equal successive times, are in geometric progression. For, if c, d, e, f, &c. be the velocities at the beginnings of those times, they are by the Proposition in geometric progression. Hence also their differences, $c - d$, $d - e$, $e - f$, &c. are in geometric. But by the last Proposition, the spaces AB, BC, CD, &c. are proportional to these differences; therefore these spaces are in geometric progression.

The Corollary may be illustrated by the following process.

Since $\dot{x} = -\dfrac{c\dot{v}}{2mr}$, $x = \dfrac{c}{2mr} \cdot (c-v)$; that is, the first

x or $AB = \dfrac{c}{2mr} \cdot (c-d)$. Again, since $x = \dfrac{c}{2mr} \cdot$ (cor.

$-v$), assume $x = AB$; then $v = d$; $\therefore x - AB$, or $BC = \dfrac{c}{2mr} \cdot (d-e)$. In the same manner, $CD = \dfrac{c}{2mr} \cdot (e-f)$;

that is, the spaces AB, BC, CD, &c. are proportional to $c-d$, $d-e$, $e-f$, &c., or they are in geometric progression.

PROPOSITION.

(178.) If the retarding force vary as the vel.|^2, and times be taken in geometric progression increasing, the velocities at the beginning of those several times are in the same geometric progression inverse; and the spaces described in these times are in arithmetic progression. (NEWTON, Lib. II. Prop. 5.)

Here $c^2 : v^2 :: r : \dfrac{rv^2}{c^2}$ = the retarding force corre-

sponding to the velocity v. Hence, $v\dot{v} = -\dfrac{2mr}{c^2} \times v^2 \dot{x}$;

$\therefore \dot{x} = -\dfrac{c^2}{2mr} \times \dfrac{\dot{v}}{v}$, and $x = \dfrac{-c^2}{2mr} \times$ hyp. log. $v +$ corr. $=$

$\dfrac{c^2}{2mr} \times$ hyp. log. $\dfrac{c}{v}$.

Also $\dot{T} = \dfrac{\dot{x}}{v} = -\dfrac{c^2}{2mr} \times \dfrac{\dot{v}}{v^2}$, and $T = \dfrac{c^2}{2mr} \times \dfrac{1}{v} +$ corr. $=$

$\dfrac{c^2}{2mr}\left(\dfrac{1}{v} - \dfrac{1}{c}\right)$. That is, if for v we write d, e, f, &c.,

the time through the space AB (Fig. Art. 177.) $= \dfrac{c^2}{2mr}$.

$\left(\frac{1}{d}-\frac{1}{c}\right)$; $T, BC = \frac{c^2}{2mr}\cdot\left(\frac{1}{e}-\frac{1}{d}\right)$; $T, CD = \frac{c^2}{2mr}\cdot\left(\frac{1}{f}-\frac{1}{e}\right)$

&c.; therefore, since the times are in geometric pro-gression, $\frac{1}{d}-\frac{1}{c}$ and $\frac{1}{e}-\frac{1}{d}$, &c. are also; $\therefore \frac{1}{c}, \frac{1}{d}, \frac{1}{e}$, &c. are in geometric progression: that is, the velocities at the beginning of the several times are inversely in the same geometric progression as the times.

Also $x = \frac{c^2}{2mr} \times \text{hyp. log.} \frac{c}{v}$; $\therefore$ the spaces in these times are in arithmetic progression.

This Proposition coincides in part with Art. 174.

PROPOSITION.

(179.) Let a sphere be projected, as in the former cases, in a resisting medium, whose retarding force varies as the $\overline{\text{vel.}}^n$; to find the velocity acquired through a given space, and the time of describing it.

Here $c^n : v^n :: r : \frac{rv^n}{c^n}$, the retarding force corre-sponding to the velocity v. Hence, $v\dot{v} = -\frac{2mr}{c^n} \times v^n \dot{x}$;

let $\frac{1}{b} = \frac{2mr}{c^n}$, then $v\dot{v} = -\frac{v^n \dot{x}}{b}$; $\therefore \dot{x} = -bv^{1-n}\dot{v}$;

and $x = -\frac{bv^{2-n}}{2-n} + \text{corr.}$ Let $x = 0$; then $v = c$, and

$x = \frac{b}{2-n}\cdot(c^{2-n} - v^{2-n})$; $\therefore v = \left(\frac{bc^{2-n}-(2-n)x}{b}\right)^{\frac{1}{2-n}}$.

Also $\dot{T} = \frac{\dot{x}}{v} = -\frac{bv^{1-n}\dot{v}}{v} = -bv^{-n}\dot{v}$; $\therefore T = -\frac{bv^{1-n}}{1-n} +$

corr.; and the correct time $= \frac{b}{1-n}(c^{1-n} - v^{1-n})$.

Cor. 1. To find the space described before the velocity is wholly destroyed, assume $v = 0$; then $x = \dfrac{b}{2-n} \times c^{2-n}$.

Cor. 2. The time, which elapses before the velocity is wholly destroyed $= \dfrac{b}{1-n} \times c^{1-n}$.

Cor. 3. If $x = 2$, it coincides with Art. 178.

PROPOSITION.

(180.) If the retarding forces vary as $\overline{\text{vel.}}|^2$, and times be taken proportional to the first velocities directly, and the first retarding forces inversely, the velocities lost will be proportional to the whole, and the spaces described will be proportional to the times and the first velocities jointly. (NEWTON, Lib. II. Prop. 7.)

The retarding force corresponding to the velocity $v = \dfrac{rv^2}{c^2}$; $\therefore v\dot{v} = -\dfrac{rv^2\dot{x}}{c^2}$; and $\dot{x} = -\dfrac{c^2\dot{v}}{rv}$. Hence, $\dot{T} = \dfrac{\dot{x}}{v} = -\dfrac{c^2\dot{v}}{rv^2}$, and $T = \dfrac{c^2}{rv} + \text{corr.} = \dfrac{c^2}{rv} - \dfrac{c}{r} = \dfrac{c}{r} \times \dfrac{c-v}{v}$. But by the Proposition, $T \propto \dfrac{c}{r}$; $\therefore v \propto c-v$, and $c-v \propto c$, or the velocity lost is proportional to the whole; and it is evident also that $c \propto v$.

Again, $\dot{T} = \dfrac{\dot{x}}{v}$; $\therefore \dot{x} = v \times \dot{T} \propto c \times \dot{T}$; $\therefore x \propto c \times T$.

SECTION II.

(181.) In the last Section, the bodies were supposed to be influenced by no forces except those of projection

and resistance ; in which case the velocity is continually retarded. In this Section they are supposed to be acted upon by gravity, or by some force tending to a center. This combined with the resistance may produce either an accelerated, or a retarded, or even a uniform velocity. The forces are supposed to act in the direction of the body's first motion, and the motion is consequently rectilinear.

PROPOSITION.

(182.) Let a spherical body descend in a fluid from rest by the action of gravity, and let the specific gravity of the sphere be to that of the fluid as $n : 1$; to find the greatest velocity which the sphere can acquire.

The same assumption being made, as in Art. 172, the difference of the weights of the sphere and an equal bulk of the fluid, or the absolute force of the body's descent $=$ $\dfrac{p\,d^3 n}{6} - \dfrac{p\,d^3}{6}$; that is the same, whatever be the velocity of the globe, the fluid being infinitely compressed. Now, let z be the space due to the velocity at any point of the descent; the resistance opposed on this account $= \dfrac{z\,p\,d^2}{8}$

(Art. 170.) ; $\therefore$ the whole resistance $= \dfrac{p\,d^3 n}{6} - \dfrac{p\,d^3}{6} - \dfrac{z\,p\,d^2}{8}$.

Divide this by $\dfrac{p\,d^3 n}{6}$ the mass moved, and the accelerating force $= 1 - \dfrac{1}{n} - \dfrac{3\,z}{4\,n\,d}$; but when the velocity is a maximum, the accelerating force $= 0$; $\therefore 1 - \dfrac{1}{n} - \dfrac{3\,z}{4\,n\,d} = 0$.

Hence, $z = \dfrac{4\,d \times (n-1)}{3}$; and $v = \sqrt{4\,m\,z} = \left(\dfrac{16\,m\,d\,.\,(n-1)}{3} \right)^{\frac{1}{2}}$.

PROPOSITION.

(183.) The greatest velocity which can be acquired by a spherical body descending in a fluid, is equal to that which the body would acquire in vacuo by its comparative gravity in falling through a space, which is to $\frac{4d}{3}$ :: the density of the sphere : the density of the fluid. (NEWTON, Lib. II. Prop. 38. Cor. 2.) For take $x : \frac{4d}{3} :: n : 1$; then,

$x = \frac{4nd}{3}$. Also $\frac{p d^3 n}{6} - \frac{p d^3}{6}$, divided by $\frac{p d^3 n}{6}$, or $1 - \frac{1}{n}$,

or $\frac{n-1}{n} =$ the force of the sphere's acceleration in the fluid, or the force of its comparative gravity. And the velocity acquired through $\frac{4nd}{3}$ by the action of this

force in vacuo $= \left(\frac{16 mnd}{3} \times \frac{n-1}{n} \right)^{\frac{1}{2}} = \left(\frac{16 md .(n-1)}{3} \right)^{\frac{1}{2}}$.

PROPOSITION.

(184.) Let a spherical body descend in a fluid from rest, and let the specific gravity of the body be to that of the fluid as n to 1; to find the velocity at any point of the descent.

By Art. 182, the accelerating force $= 1 - \frac{1}{n} - \frac{3z}{4nd}$; hence, if $x =$ the space described, $\dot{z} = \dot{x} - \frac{\dot{x}}{n} - \frac{3 z \dot{x}}{4nd}$; $\therefore \frac{\dot{x}}{4nd} =$

$\frac{\dot{z}}{4nd - 4d - 3z} = \frac{\dot{z}}{4d.(n-1) - 3z}$, and $\frac{x}{4nd} = -\frac{1}{3} \times$ hyp.

log. $\left(4d.(n-1) - 3z \right) +$ corr. Now let $x = 0$, then $z = 0$;

$\therefore c = \frac{1}{3} \times$ hyp. log. $4d.(n - 1)$; consequently $\frac{x}{4nd} =$

$-\frac{1}{3} \times$ hyp. log. $\dfrac{4d.(n-1)-3z}{4d.(n-1)}$. Let $e =$ the number whose

hyp. log. is 1 ; then $e^{\frac{-3z}{4nd}} = \dfrac{4d.(n-1)-3z}{4d.(n-1)} = 1 - \dfrac{3z}{4d.(n-1)}$;

$\therefore z = \dfrac{4d}{3}.(n-1).\left(1 - e^{\frac{-3z}{4nd}}\right)$; and the velocity, or

$\sqrt{4mz} = \left(\dfrac{16md.(n-1)}{3}\right)^{\frac{1}{2}} . \left(1 - e^{\frac{-3z}{4nd}}\right)^{\frac{1}{2}}$.

Cor.   If x be increased *sine limite*, $e^{\frac{-3z}{4nd}}$ will be a quantity indefinitely small; and the ultimate velocity of the sphere will $= \left(\dfrac{16md.(n-1)}{3}\right)^{\frac{1}{2}}$, as in Art. 182.

Proposition.

(185.)   If a spherical body of given diameter be immersed in a fluid, and its specific gravity be indefinitely less than that of the fluid, the velocity of ascent will be uniform, and equal to that which a heavy body would acquire in falling from rest through $\frac{4}{3}$ of the diameter by the action of gravity.

Suppose the sphere to have ascended from rest through a space $= x$, and let z be the space due to the velocity at that point; then, if $n =$ the specific gravity of the sphere, its weights $= \dfrac{pd^3 n}{6}$, and the resistance to its motion arising from the velocity $= \dfrac{zpd^2}{8}$; $\therefore$ the whole resistance opposed to the sphere's ascent $= \dfrac{pd^3 n}{6} + \dfrac{zpd^2}{8}$.   But its force of ascent is the weight of a quantity of fluid equal in magnitude to the sphere, and this weight $= \dfrac{pd^3}{6}$, the

specific gravity being 1. Hence the force of the sphere's
ascent $=\dfrac{p d^3}{6}-\dfrac{p d^3 n}{6}-\dfrac{z p d^3}{8}$; divide this by the sphere's

weight, and we get the accelerating force $=\dfrac{1}{n}-1-\dfrac{3 z}{4 n d}=$

$\dfrac{1}{n}-\dfrac{3 z}{4 n d}$, for 1 vanishes in respect of the other terms. Hence,

$\ddot{z}=\dfrac{\dot{x}}{n}-\dfrac{3 z \dot{x}}{4 n d}$, and $\dfrac{\dot{x}}{4 n d}=\dfrac{\dot{z}}{4 d-3 z}$; $\therefore \dfrac{x}{4 n d}=-\dfrac{1}{3}\times$ hyp.
log. of $(4 d-3 z)+$ corr. Let $x=0$, then $z=0$; $\therefore$ corr. $=$
$\dfrac{1}{3}$ hyp. log. $4 d$; $\therefore \dfrac{x}{4 n d}=-\dfrac{1}{3}\times$ hyp. log. $\dfrac{4 d-3 z}{4 d}$.

Hence, assuming e as before, the number whose hyper-
bolic logarithm is 1, we have $\dfrac{4 d-3 z}{4 d}=e^{\frac{-3 x}{4 n d}}=0$, for

$\dfrac{3 x}{4 n d}$ is indefinitely great; $\therefore z=\dfrac{4 d}{3}$; and the velocity at

this point $=\sqrt{4 m z}=\sqrt{\dfrac{16 m d}{3}}$, which is the same

with that acquired by gravity through $\dfrac{4 d}{3}$.

As this reasoning is equally true for all points of the
ascent, the velocity of the body is uniform.

COR. Spherical bodies without weight, of different
diameters, ascend in fluids with uniform velocities, which
vary as the square root of the diameters.

PROPOSITION.

(186.) Let a body descend or be projected in a re-
sisting medium directly toward a center of force, and be
attracted by a constant force toward the center; to find
the greatest velocity which it can acquire, the retarding

force of the medium being supposed to vary as the velocity.

Let F represent the constant force; $r=$ the retarding force corresponding to the velocity c, and v any variable velocity. Then the retarding force corresponding to $v = \dfrac{r\,v}{c}$; $\therefore$ the whole force of acceleration toward the center $= F - \dfrac{r v}{c}$. But when the velocity is a maximum, the force of acceleration $= 0$; $\therefore F - \dfrac{r v}{c} = 0$, and $v = \dfrac{c\,F}{r}$.

Cor. The greatest velocity v : the velocity c :: F : the resistance r corresponding to c. (Newton, Lib. II. Prop. 3. Cor. 1.)

Proposition.

(187.) Let a body descend or be projected in a resisting medium under the same circumstances as in the last case, toward the center of force. To find the space and time corresponding to any velocity. (Newton, Lib. II. Prop. 3.)

The accelerating force upon the descending body, by the last Proposition, $= F - \dfrac{r\,v}{c} = F - av$, where $a = \dfrac{r}{c}$.

Hence, if x be the space described, $v\dot{v} = 2m\dot{x} \cdot (F - av)$;

$\therefore \dot{x} = \dfrac{v\dot{v}}{2m.(F - av)} = \dfrac{v\dot{v}}{2ma.\left(\dfrac{F}{a} - v\right)}$. Let $\dfrac{F}{a} = b$; then

$\dot{x} = \dfrac{v\dot{v}}{2ma.(b - v)} = \dfrac{1}{2ma} \times \left(\dot{v} + \dfrac{v\dot{v}}{b - v} - \dot{v}\right) = \dfrac{1}{2ma} \times$

$\left(\dfrac{b\dot{v}}{b - v} - \dot{v},\right)$ and $x = \dfrac{1}{2ma} \times \left(-b \times \text{h. log. } (b - v) - v\right)$

+corr. Let $x=0$; then, if the body descend from rest, $v=0$; and x corrected $= \dfrac{b}{2ma} \times$ hyp. log. $\dfrac{b}{b-v} - \dfrac{1}{2ma} \times v$. But if it be projected, let the velocity of pro-jection $=c$; then $x = \dfrac{b}{2ma} \times$ hyp. log. $\dfrac{b-c}{b-v} + \dfrac{1}{2ma} \cdot (c-v)$.

Also, $\dot{T} = \dfrac{\dot{x}}{v} = \dfrac{\dot{v}}{2ma \cdot (b-v)}$; $\therefore T = -\dfrac{1}{2ma} \times$ hyp. log. $(b-v) + $ corr.; therefore, if the body descend from rest, the correct time $= \dfrac{1}{2ma} \times$ hyp. log. $\dfrac{b}{b-v}$. If it be pro-jected with the velocity c, $T = \dfrac{1}{2ma} \times$ hyp. log. $\dfrac{b-c}{b-v}$.

PROPOSITION.

(188.) The same supposition remaining, let the body be projected directly from the center of force. To find the space and time corresponding to any velocity.

In this case, the force of retardation $= F + \dfrac{rv}{c}$, and $v\dot{v} = -2m\dot{x}.\left(F + \dfrac{rv}{c}\right) = -2m\dot{x}.(F+av)$, and $\dot{x} = \dfrac{1}{2m} \times \dfrac{-v\dot{v}}{F+av.} = \dfrac{1}{2ma} \times \dfrac{-v\dot{v}}{b+v}$; and, as in the last Proposition,

$x = \dfrac{b}{2ma} \times$ hyp. log. $\dfrac{b+v}{b+c} + \dfrac{1}{2ma} \cdot (c-v)$.

Also $\dot{T} = \dfrac{\dot{x}}{v} = -\dfrac{1}{2ma} \times \dfrac{\dot{v}}{b+v}$; $\therefore T = -\dfrac{1}{2ma} \times$ hyp. log. $(b+v) + \dfrac{1}{2ma} \times$ hyp. log. $(b+c) = \dfrac{1}{2ma} \times$ hyp. log. $\dfrac{b+c}{b+v}$.

P

PROPOSITION.

(189.) Let a body descend or be projected toward a center as before, the retarding force of the medium varying as the square of the velocity. To find the greatest velocity which the body can acquire.

The retarding force corresponding to the velocity v is in this case $\dfrac{r v^2}{c^2}$; $\therefore$ the whole force of acceleration $=$

$F - \dfrac{r v^2}{c^2}$. But when the velocity is a maximum, this

force $= 0$; $\therefore v = c \times \sqrt{\dfrac{F}{r}}$.

Cor. 1. $V : c :: \sqrt{F} : \sqrt{r}$. (Newton, Lib. II. Prop. 8. Cor. 3.)

Cor. 2. Since $r = \dfrac{3 c^2}{16 m n d}$ (Art. 172.), if F represent

the comparative gravity of the body, or $= \dfrac{n-1}{n}$, $v = c \times$

$\left(\dfrac{n-1}{n} \times \dfrac{16 m n d}{3 c^2}\right)^{\frac{1}{2}} = \left(16 m d \times \dfrac{n-1}{3}\right)^{\frac{1}{2}}$; and this Proposition coincides with Art. 182.

PROPOSITION.

(190.) Let a body be projected in a resisting medium, directly towards a center of force, under the same supposition as in the last case. To find the space and time corresponding to any velocity.

The accelerating force in this case corresponding to the velocity $v = F - \dfrac{r v^2}{c^2} = F - a v^2$, if $a = \dfrac{r}{c^2}$; $\therefore v \dot{v} = 2 m \dot{x}$.

$(F - a v^2)$, and $\dot{x} = \dfrac{v \dot{v}}{2 m.(F - a v^2)} = \dfrac{1}{2 m a} \times \dfrac{v \dot{v}}{b^2 - v^2}$, where

$b^2 = \dfrac{F}{a}$; $\therefore x = -\dfrac{1}{4\,m\,a} \times$ hyp. log. $(b^2 - v^2) +$ corr. **When**

$x = 0$, let $v = c$; $\therefore x = \dfrac{1}{4\,m\,a} \times$ hyp. log. $\dfrac{b^2 - c^2}{b^2 - v^2}$.

Also $T' = \dfrac{\dot{x}}{v} = \dfrac{\dot{v}}{2\,m\,a.(b^2 - v^2)}$; $\therefore T = \dfrac{1}{4\,m\,a\,b} \times$ hyp. log.

$\dfrac{b + v}{b - v} +$ corr. When $T = 0$, let $v = c$; $\therefore T = \dfrac{1}{4\,m\,a\,b} \times$

$\left(\text{hyp. log. } \dfrac{b + v}{b - v} - \text{hyp. log. } \dfrac{b + c}{b - c}\right)$.

If the body descend from rest, in correcting the fluents, v must be assumed $= \dot{0}$.

(191.) The first part of this Proposition will coincide with Art. 184, by taking the fluent without correction, and substituting the proper values for a and b, and F and r.

Thus, $\dot{x} = \dfrac{1}{2\,m\,a} \times \dfrac{v\dot{v}}{b^2 - v^2}$; $\therefore x = -\dfrac{1}{4\,m\,a} \times$ hyp. log.

$(b^2 - v^2) +$ corr. Let $x = 0$, and $v = 0$, or the body descend by the force F, its comparative gravity; then, $x = -\dfrac{1}{4\,m\,a} \times$ hyp. log. $\dfrac{b^2 - v^2}{b^2}$. Hence, if e be the number

whose hyperbolic logarithm is 1, we have $e^{-4mx} = \dfrac{b^2 - v^2}{b^2}$.

Now for a write $\dfrac{r}{c^2}$, or $\dfrac{3\,c^2}{16\,m\,n\,d} \times \dfrac{1}{c^2}$ (Art. 172.), or

$\dfrac{3}{16\,m\,n\,d}$; for F write $\dfrac{n-1}{n}$, and for b^2 the fraction $\dfrac{F}{a}$;

then v^2 will $= 16\,m\,d \times \dfrac{n-1}{3} . \left(1 - e^{\frac{-3x}{4md}}\right)$; and $v =$

$\left(\dfrac{16\,m\,d.(n-1)}{3}\right)^{\frac{1}{2}} . \left(1 - e^{\frac{-3x}{4md}}\right)^{\frac{1}{2}}$.

PROPOSITION.

(192.) The same supposition remaining, let the body be projected directly from the center of force. To find the space and time corresponding to any velocity.

The retarding force in this case $= F + \dfrac{r v^2}{c^2}$, and $v \dot{v} =$

$- 2 m \dot{x} . \left(F + \dfrac{r v^2}{c^2} \right) = - 2 m \dot{x} . (F + a v^2);$ $\therefore \dot{x} = -$

$\dfrac{1}{2 m} \times \dfrac{v \dot{v}}{F + a v^2} = - \dfrac{1}{2 m a} \times \dfrac{v \dot{v}}{b^2 + v^2};$ and $x = - \dfrac{1}{4 m a} \times$ hyp.

log. $(b^2 + v^2) + \text{corr.} = \dfrac{1}{4 m a} \times$ hyp. log. $\dfrac{b^2 + c^2}{b^2 + v^2}$, where c

is the velocity of projection.

Also, $T = \dfrac{\dot{x}}{v} = - \dfrac{1}{2 m a} \times \dfrac{\dot{v}}{b^2 + v^2};$ $\therefore T = - \dfrac{1}{2 m a b^2} \times$ a

circular arc, whose radius is b, and tangent $v + \text{corr.}$ Let this arc $= N$. Now if $T = 0$, $v = c$. Let the arc, whose radius is b, and tangent $c = M$; then T corrected $=$

$\dfrac{1}{2 m a b^2} . (M - N).$

Cor. 1. To find the greatest height to which the body will rise, take $v = 0$; then $x = \dfrac{1}{4 m b} \times$ hyp. log. $\dfrac{b^2 + c^2}{b^2}$

$=$ (by substituting the values of a and b^2, as in the Pro-

position,) $\dfrac{c^2}{4 m r} \times$ hyp. log. $\dfrac{F + r}{F}$.

Cor. 2. To find the time, in which the body will lose all its velocity, assume $v = 0$; then the arc $N = 0$;

$\therefore T = \dfrac{1}{2 m a b^2} \times M.$

Cor. 3. Since the time in which the body loses its

whole velocity varies as M, it varies as $b \times M$; or as the sector of a circle, whose radius is b, and tangent c. (NEWTON, Lib. II. Prop. 9.)

COR. 4. And the time in which the velocity is diminished by the quantity $c - v$ varies as $M - N$, or as $b \times (M - N)$; that is, it is proportional to the difference of two sectors, whose tangents arc c and v, to the same given radius b.

PROPOSITION.

(193.) The same supposition being made, if spaces be taken in arithmetic progression, the whole force which accelerates or retards, according as the body descends or ascends, will be in geometric progression. (NEWTON, Lib. II. Prop. 8.)

1. By Art. 190, if the body be projected downward with a velocity c, $x = \dfrac{1}{4ma} \times$ hyp. log. $\dfrac{b^2 - c^2}{b^2 - v^2}$; if it descend from rest, $x = \dfrac{1}{4ma} \times$ hyp. log. $\dfrac{b^2}{b^2 - v^2}$. In either case, if x be in arithmetic progression, the fractional part of the expression is in geometric. Hence, since the numerator is constant, $b^2 - v^2$, or $\dfrac{F}{a} - v^2$, or $F - av^2$, or its equal $F - \dfrac{rv^2}{c^2}$; the whole force of acceleration is also in geometric progression.

2. Let the body be projected from the center, or ascend. Here (by Art. 192.) $x = \dfrac{1}{4ma} \times$ hyp. log. of $\dfrac{b^2 + c^2}{b^2 + v^2}$; therefore, if x be in arithmetic progression, $b^2 + v^2$, or $\dfrac{F}{a} + v^2$, or $F + \dfrac{rv^2}{c^2}$, the whole of the retarding force is in geometric progression.

PROPOSITION.

(194.) In general, let a body be projected with a given velocity in a resisting medium directly to, or directly from, a center of force; and let it be attracted by a constant force to that center. To find the space and time corresponding to any velocity, the retarding force being supposed to vary as the $\overline{\text{vel.}}|^n$.

1. If the body descend, the whole force of acceleration

$$= F - \frac{rv^n}{c^n} \; ; \; \therefore \; v\dot{v} = 2m\dot{x} \cdot \left(F - \frac{rv^n}{c^n} \right), \text{ and } \dot{x} =$$

$$\frac{v\dot{v}}{2m \cdot \left(F - \frac{rv^n}{c^n} \right)} ;$$

Also, $\dot{T} = \dfrac{\dot{x}}{v} = \dfrac{\dot{v}}{2\,m.\left(F - \dfrac{rv^n}{c^n} \right)}$.

2. If a body ascend, the whole force of retardation $=$

$$F + \frac{rv^n}{c^n} \; ; \; \text{ and } \; v\dot{v} = -\,2m\dot{x}.\left(F + \frac{rv^n}{c^n} \right), \text{ and } \dot{x} =$$

$$\frac{-v\dot{v}}{2m.\left(F + \frac{rv^n}{c^n} \right)} \text{ .}$$

Also, $\dot{T} = \dfrac{\dot{x}}{v} = \dfrac{-\dot{v}}{2\,m.\left(F + \dfrac{rv^n}{c^n} \right)}$.

The fluents must be found for the particular cases.

(195.) These expressions may be applied to the descent of bodies in resisting mediums at the surface of the earth, by substituting for F the fraction $\dfrac{n-1}{n}$, which represents the force of its comparative gravity. An instance of this is given in Art. 189.

SECTION III.

PROPOSITION.

(196.) To determine the resistance of a medium, by which a body when impelled by a given force F, acting in parallel lines, may describe a given curve.

Let ABE be the given curve; BS the direction of the force at B. Draw BP perpendicular to BS, meeting the axis AS, or some given line parallel to BS, in P. Take $BQ=s=\frac{1}{4}$ the chord of curvature at B. Let $AP=x$, $BD=\dot{x}$, $AB=z$, $BC=\dot{z}$; and draw DR perpendicular to BC.
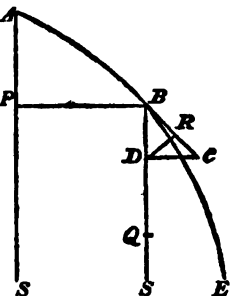
Now a body must fall through $\frac{1}{4}$th of the chord of curvature by the constant force F, to acquire the velocity in the curve, whether it moves in vacuo, or in a resisting medium. For the resistance has no effect in causing the body to deviate from the tangent; it only retards its motion in such a manner, that it may always bear a just proportion to the given force F. Hence, $v = \sqrt{2mFs}$, and $\dot{v} = \dfrac{mF\dot{s}}{\sqrt{2mFs}} =$ the whole increase of velocity in the direction BC, in the time of describing BC, or in a time $= \dfrac{\dot{z}}{v}$. Also, the increase in the direction BS by the force F in that time $= 2mF\dot{t} = \dfrac{2mF\dot{z}}{\sqrt{2mFs}}$; and this increase : the increase of velocity in the direction BC from that cause :: $BD : BR :: BC : BD :: \dot{z} : \dot{x}$;

$\therefore$ the increase in the direction $BC = \dfrac{2mF\dot{z}}{\sqrt{2mFs}} \times \dfrac{\dot{x}}{\dot{z}} =$

$\dfrac{2mF\dot{x}}{\sqrt{2mFs}}$; hence, the effect of the resistance $= m \times$

$\dfrac{F\dot{s} - 2F\dot{x}}{\sqrt{2mFs}}$, the difference of these increments; or the decrement of velocity arising from the resistance $= -m \times$ $\dfrac{F\dot{s} - 2F\dot{x}}{\sqrt{2mFs}}$. Therefore, since $F \propto$ the increment or decrement of velocity generated in a given time, the resistance : the force $F :: -m \times \dfrac{F\dot{s} - 2F\dot{x}}{\sqrt{2mFs}} : \dfrac{2mF\dot{z}}{\sqrt{2mFs}}$

$$:: \dfrac{\dot{s} - 2\dot{x}}{2\dot{z}} : 1,$$ by omitting the negative sign, which denotes a retarding force.

(197.) To obtain an expression in terms of the abscissa and the curve. Let $y = PB$; then $\dot{s}^2 = \dot{x}^2 + \dot{y}^2$, and $s = \dfrac{\dot{z}^2}{\ddot{x}}$ (Art. 100.) $= \dfrac{\dot{x}^2 + \dot{y}^2}{\ddot{x}}$; therefore, if $\dot{y}$ be assumed constant, $\dot{s} = \dfrac{2\dot{x}\ddot{x}^2 - (\dot{x}^2 + \dot{y}^2) \times \dddot{x}}{\ddot{x}^2} = \dfrac{2\dot{x}\ddot{x}^2 - \dot{s}^2\dddot{x}}{\ddot{x}^2}$;

hence, $\dfrac{\dot{s} - 2\dot{x}}{2\dot{z}} = -\dfrac{\dot{z}\dddot{x}}{2\ddot{x}}$. Therefore, the resistance : the force $F :: \dfrac{\dot{z}\dddot{x}}{2\ddot{x}^2} : 1$, the negative sign being omitted as before.

PROPOSITION.

(198.) To determine the resistance by which a body may describe any given curve about a center of force.

In this case, F is considered as variable. Hence, by

making the same supposition as before, since $v = \sqrt{2mFs}$,

$$\dot{v} = m \times \frac{F\dot{s} + s\dot{F}}{\sqrt{2mFs}} = \text{ the whole increase of velocity in a}$$

tangential direction, during the time $\dfrac{\dot{z}}{\sqrt{2mFs}}$.   Also,

the increase in the same direction by the force F in the

same time $= \dfrac{2mF\dot{x}}{\sqrt{2mFs}}$, as before; or if $SB = w$, since $\dot{x} =$

$-\dot{w}$, this increase $= -\dfrac{2mF\dot{w}}{\sqrt{2mFs}}$; therefore the decre-

ment of velocity, arising from the resistance, $= -m \times$

$\dfrac{F\dot{s} + s\dot{F} + 2F\dot{w}}{\sqrt{2mFs}}$; and this resistance : the force F :: $m \times$

$$\frac{F\dot{s} + s\dot{F} + 2F\dot{w}}{\sqrt{2mFs}} : \frac{2mF\ddot{z}}{\sqrt{2mFs}} :: \frac{F\dot{s} + s\dot{F} + 2F\dot{w}}{2F\ddot{z}} : 1, \text{ the}$$

negative sign being again omitted.

Cor.   If the force act in parallel lines, or S be at an infinite distance, F is constant, and $\dot{F} = 0$; therefore, in this case, the resistance : F :: $\dfrac{F\dot{s} + 2F\dot{w}}{2F\ddot{z}} : 1 :: \dfrac{\dot{s} + 2\dot{w}}{2\ddot{z}} : 1$

:: $\dfrac{\dot{s} - 2\dot{x}}{2\ddot{z}}$: 1, the same proportion which was found

in Art. 196.

<div align="center">EXAMPLES.</div>

(199.) Ex. 1. Let the curve be the common parabola, and the force act in lines parallel to *AS*.

Here $ax = y^2$; $\therefore a\dot{x} = 2y\dot{y}$; and $a\ddot{x} = 2\dot{y}^2 = 2$, if $\dot{y}$ be assumed constant, and $= 1$. Hence, $\ddot{x} = 0$; also, $\dot{z} =$

$$\frac{(4y^2+a^2)^{\frac{1}{2}}}{a} \ (\text{Art. 55.}) = \left(\frac{4x+a}{a}\right)^{\frac{1}{2}} ; \ \therefore \frac{\dot{z}\ddot{x}}{2\ddot{x}} = \frac{\left(\frac{4x+a}{a}\right)^{\frac{1}{2}} . 0}{2\ddot{x}^2},$$

or the resistance is nothing.

Ex. 2. Let the curve be any parabola, whose equation is $a^{n-1}x = y^n$, the force acting in parallel lines as before.

Here $\dot{x} = \frac{ny^{n-1}\dot{y}}{a^{n-1}}$; therefore, $\ddot{x} = \frac{n.(n-1).y^{n-2}\dot{y}^2}{a^{n-1}}$; and

$$\overset{...}{x} = \frac{n.(n-1).(n-2).y^{n-3}\dot{y}^3}{a^{n-1}} \ . \ \text{Also,} \ \dot{z} = \frac{(n^2y^{2n-2}+a^{2n-2})^{\frac{1}{2}}.\dot{y}}{a^{n-1}} ;$$

therefore if $\dot{y}=1$, the resistance, or $\frac{\dot{z}\ddot{x}}{2\overset{...}{x}}$, $= \frac{n-2}{2\,n.(n-1)}$.

$\frac{(n^2y^{2n-2}+a^{2n-2})^{\frac{1}{2}}}{y^{n-1}}$, F being assumed $= 1$.

Cor. 1. If $n = 2$, the resistance $= 0$.

Cor. 2. If n be greater than 1, but less than 2, the expression becomes negative; or the medium propels the body, instead of retarding it.

Cor. 3. If $n = 1$, the expression becomes negative, and infinite; that is, the propelling force of the medium is infinite, and the body moves in a right line.

Ex. 3. Let the curve be a cycloid, and the force act in lines parallel to the axis AM.

Let $AM=a$. The perpendicular $BQ=\frac{1}{2}s$ therefore $\dot{x} = -\frac{1}{2}\dot{s}$, and $\dot{s} = -2\dot{x}$.
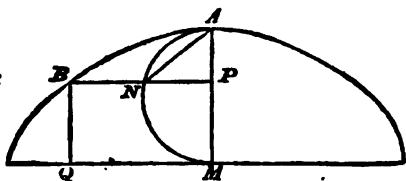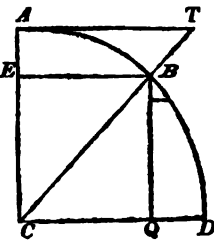
Also, $\dot{z} : \dot{x} :: AN : AP :: AM : AN :: \sqrt{AM} : \sqrt{AP}$

$:: a^{\frac{1}{2}} : x^{\frac{1}{2}} ; \ \therefore \dot{z} = \frac{a^{\frac{1}{2}}\dot{x}}{x^{\frac{1}{2}}}$; hence the resistance, or $\frac{\dot{s}-2\dot{x}}{2\dot{z}}$, $=$

$$-\frac{4\dot{x}}{2\dot{z}} = -\frac{4\dot{x}}{2} \times \frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}\dot{x}} = -\frac{2x^{\frac{1}{2}}}{a^{\frac{1}{2}}} = -2 \times \sqrt{\frac{AP}{AM}} = -\frac{2AN}{AM} =$$

$\dfrac{2AN}{AM}$, the negative sign being omitted, and the force of

gravity being assumed $= 1$. The velocity $\propto \sqrt{BQ}$.

Ex. 4. Let the curve be a quadrant of a circle, the force as before.

Take $AE = x$, $AC = a$, $BQ = s$;

then $s = a - x$; $\therefore \dot{s} = -\dot{x}$, and $\dfrac{\dot{s} - 2\dot{x}}{2\dot{z}}$

$= -\dfrac{3\dot{x}}{2\dot{z}}$. But $\dot{x} : \dot{z} :: BE : BC$;

$\therefore -\dfrac{3\dot{x}}{2\dot{z}} = -\dfrac{3BE}{2BC}$, or changing the

sign, the resistance $= \dfrac{3BE}{2BC}$, gravity being assumed $= 1$.

Cor. 1. At A, BE vanishes; therefore the resistance $= 0$. At C, $BE = BC$; and the resistance : gravity :: 3 : 2. When $3BE = 2BC$, the resistance $= 1$, $=$ gravity.

Cor. 2. The velocity at $B \propto \sqrt{BQ}$; also the resistance varies as BE. Hence, if the resistance be supposed to vary as $V^2 \times$ the density of the medium, $BE \propto BQ \times$ the density; therefore, the density at $B \propto \dfrac{BE}{BQ} \propto \dfrac{BE}{CE} \propto$

$\dfrac{AT}{AC} \propto$ the tangent of the arc AB. (Newton, Lib. II. Prop. 10.)

Ex. 5. Let the force tend to the center of a logarithmic spiral, and vary as the $\overline{\text{dist.}}^n$.

Here $F \propto w^n$; $\therefore \dot{F} \propto nw^{n-1}\dot{w}$; and the expression

$\dfrac{F\dot{s} + s\ddot{F} + 2F\dot{w}}{2F\dot{z}}$ for the resistance, since $s = w$, becomes

$\dfrac{w^n\dot{w} + nw^n\dot{w} + 2w^n\dot{w}}{2w^n\dot{z}} = \dfrac{n+3}{2} \times \dfrac{\dot{w}}{\dot{z}}$. But $\dot{w} : \dot{z}$ in a con-

stant ratio, $:: a : b$; $\therefore \dfrac{\dot{w}}{\dot{z}} = \dfrac{a}{b}$; and the expression for

the resistance is $\dfrac{n+3}{2} \times \dfrac{a}{b}$, the force tending to S being

assumed $= 1$.

Cor. 1.   If $n = -3$, or the force varies inversely as the cube of the distance, the resistance $= 0$.

Cor. 2.   If n be a greater negative number than 3, the medium propels the body.

Cor. 3.   If for F we substitute w^n, then the resist-

ance : the force $w^n :: \dfrac{n+3}{2} \times \dfrac{a}{b}$: 1; therefore the re-

sistance $= \dfrac{n+3}{2} \times \dfrac{a}{b} \times w^n$.

Cor. 4.   Since the velocity $\propto \sqrt{Fs}$, or as $w^{\frac{n+1}{2}}$, and the density of the medium varies as the resistance divided by the square of the velocity; the density in this case $\propto$

$\dfrac{w^n}{w^{n+1}} \propto \dfrac{1}{w}$; therefore, conversely, if the density of the

medium $\propto \dfrac{1}{w}$, the body may describe the logarithmic

spiral, whatever be the value of n.   (NEWTON, Lib. II. Prop. 16.)

Cor. 5.   If $n = -2$, then the force varies as $\dfrac{1}{w^3}$; or F

varies as the square of the density.   (NEWTON, Lib. II. Prop. 15.)

CHAP. XXI.

FLUENTS.

SECTION I.

(200.) METHODS have been already proposed for finding, in certain cases, the fluent from the fluxion. The following Chapter is intended to furnish a variety of Examples, with Rules to facilitate the operation.

RULE. The fluents of such fluxions as are of the form $\dfrac{x^m \dot{x}}{a \pm x}$, $\dfrac{x^m \dot{x}}{a^x \pm x^2}$, where m is a whole positive number; and those of the form $\dfrac{x^{rm-1} \dot{x}}{a^m \pm x^m}$, where r is a whole positive number, may be found by dividing the numerator by the denominator in an inverted order.

FLUENT 1.

(201.) 1. To find the fluent of $\dfrac{x \dot{x}}{a + x}$.

$$x + a \,) \; x \dot{x} \; \left(\, \dot{x} \right.$$
$$\underline{x \dot{x} + a \dot{x}}$$
$$\underline{- a \dot{x}}$$
$$x + a.$$

Hence, the fluent is $x - a \times$ hyp. log. $(a + x)$.

2. In the same manner, the fluent of $\dfrac{x\dot{x}}{a-x} = -x - a$ × hyp. log. $(a-x)$.

3. The fluent of $\dfrac{x\dot{x}}{x-a} = x + a \times$ hyp, log. $(x-a)$.

FLUENT 2.

(202.) 1. The fluent of $\dfrac{x^2\dot{x}}{a+x} = \dfrac{x^2}{2} - ax + a^2 \times$ hyp. log. $(a+x)$.

2. The fluent of $\dfrac{x^2\dot{x}}{a-x} = -\dfrac{x^2}{2} - ax - a^2 \times$ hyp. log. $(a-x)$.

3. The fluent of $\dfrac{x^2\dot{x}}{x-a} = \dfrac{x^2}{2} + ax + a^2 \times$ hyp. log. $(x-a)$.

FLUENT 3.

(203.) The fluent of $\dfrac{p\,x^2\dot{x}}{a-x} =$ the fluent of p ×

$$\left(-x^2\dot{x} - ax\dot{x} - a^2\dot{x} + \dfrac{a^3\dot{x}}{a-x}\right) = -\dfrac{px^3}{3} - \dfrac{pax^2}{2} - pa^2x$$

$+pa^3 \times$ hyp. log. $\dfrac{a}{a-x}$; the last term being corrected by making $x=0$. This fluent occurs in the problem for finding the content of the solid, generated by the revolution of the cissoid of Diocles about its axis.

FLUENT 4.

(204.) 1. To find the fluent of $\dfrac{x^m\dot{x}}{x-a}$, where m is a whole positive number.

$$x-a\big)\ x^m \dot{x}\ \big(x^{m-1}\dot{x}+ax^{m-2}\dot{x}+\&c.$$
$$x^m \dot{x}\ -\ ax^{m-1}\dot{x}$$
$$\overline{\qquad\qquad\qquad}$$
$$ax^{m-1}\dot{x},\ \&c.$$

Continue the division, till the index of x in the remainder $=0$; there are then m terms in the quotient, the last of which is $a^{m-1}\dot{x}$, and the remainder $=\dfrac{a^m \dot{x}}{x-a}$. Hence, the whole fluent $=\dfrac{x^m}{m}+\dfrac{ax^{m-1}}{m-1}+\&c.+a^m\times$ hyp. log. $(x-a)$.

2. The series is the same for the fluent of $\dfrac{x^m \dot{x}}{x+a}$, but the signs of the terms are alternately $+$ and $-$.

3. The fluent of $\dfrac{x^m \dot{x}}{a-x}$ is found by dividing the numerator by $-x+a$; and the signs of the terms are all negative.

FLUENT 5.

(205.) 1. To find the fluent of $\dfrac{x^2 \dot{x}}{a^2+x^2}$.

$$x^2+a^2\big)\ x^2\dot{x}\ \big(\dot{x}$$
$$x^2\dot{x}\ +\ a^2\dot{x}$$
$$\overline{\qquad\qquad\qquad}$$
$$-\ a^2\dot{x}$$
$$\overline{\qquad\qquad\qquad}$$
$$a^2+x^2.$$

Hence, the fluent $=x-a$, a circular arc of radius a, and tangent x.

2. The fluent of $\dfrac{x^2\dot{x}}{a^2-x^2}=-x+\dfrac{a}{2}\times$ hyp. log. $\dfrac{a+x}{a-x}$.

3. The fluent of $\dfrac{x^2\dot{x}}{x^2-a^2}=x+\dfrac{a}{2}\times$ hyp. log. $\dfrac{x-a}{x+a}$.

4. In the same manner, the fluent of $\dfrac{x^4 \dot{x}}{a^2 \pm x^2}$, &c. can be found.

FLUENT 6.

(206.) 1. To find the fluent of $\dfrac{x^m \dot{x}}{a^2 + x^2}$, where m is an even positive number.

The numerator divided by the denominator, in an inverted order, $= x^{m-2}\dot{x} - a^2 x^{m-4}\dot{x} + a^4 x^{m-6}\dot{x} - \&c. \pm a^{m-2}\dot{x}$ $\mp \dfrac{a^m \dot{x}}{a^2 + x^2}$, the number of terms in the quotient being $\dfrac{m}{2}$, and the remainder $\mp \dfrac{a^m \dot{x}}{a^2 + x^2}$; hence, the fluent $= \dfrac{x^{m-1}}{m-1}$ $- \dfrac{a^2 x^{m-3}}{m-3} + \&c. \pm a^{m-2}x \mp a^{m-1} \times$ a circular arc of radius a, and tangent x.

2. In the same manner, the fluent of $\dfrac{x^m \dot{x}}{a^2 - x^2}$, where m is an even positive number, $= - \dfrac{x^{m-1}}{m-1} - \dfrac{a^2 x^{m-3}}{m-3} - \&c.$ $+ \dfrac{a^{m-1}}{2} \times$ hyp. log. $\dfrac{a+x}{a-x}$.

3. The fluent of $\dfrac{x^m \dot{x}}{x^2 - a^2}$, determined in the same manner, has all the terms positive; the remainder is $\dfrac{a^{m-1}}{2}$ $\times$ hyp. log. $\dfrac{x-a}{x+a}$.

FLUENT 7.

(207.) 1. To find the fluent of $\dfrac{x^3 \dot{x}}{a^2 + x^2}$.

The numerator divided by the denominator in an inverted order, gives $x\dot{x} - \dfrac{a^2 x \dot{x}}{a^2 + x^2}$; and the fluent $= \dfrac{x^2}{2} - \dfrac{a^2}{2} \times$ hyp. log. $(a^2 + x^2)$.

2. The fluent of $\dfrac{x^3 \dot{x}}{a^2 - x^2} = -\dfrac{x^2}{2} - \dfrac{a^2}{2} \times$ hyp. log. $(a^2 - x^2)$.

3. The fluent of $\dfrac{x^3 \dot{x}}{x^2 - a^2} = \dfrac{x^2}{2} + \dfrac{a^2}{2} \times$ hyp. log. $(x^2 - a^2)$.

4. In the same manner, the fluents of $\dfrac{x^5 \dot{x}}{a^2 \pm x^2}$, $\dfrac{x^7 \dot{x}}{a^2 \pm x^2}$, &c. can be found.

FLUENT 8.

(208.) 1. To find the fluent of $\dfrac{x^m \dot{x}}{a^2 + x^2}$, where m is an odd number.

The numerator divided by the denominator $x^2 + a^2$ gives $x^{m-2} \dot{x} - a^2 x^{m-4} \dot{x} + a^4 x^{m-6} \dot{x} - $ &c. $\pm a^{m-3} x \dot{x}$, the number of terms being $\dfrac{m-1}{2}$, and the remainder

$\mp \dfrac{a^{m-1} x \dot{x}}{a^2 + x^2}$; so that the fluent is $\dfrac{x^{m-1}}{m-1} - \dfrac{a^2 x^{m-3}}{m-3} +$ &c., $\pm \dfrac{a^{m-3} x^2}{2} \mp$ the remainder $\dfrac{a^{m-1}}{2} \times$ hyp. log. $(a^2 + x^2)$.

2. The fluent of $\dfrac{x^m \dot{x}}{a^2 - x^2}$ is obtained in the same manner; but the signs are all negative, and the remainder is $-\dfrac{a^{m-1}}{2} \times$ hyp. log. $(a^2 - x^2)$.

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3. The fluent of $\frac{x^m \dot{x}}{x^2 - a^2}$ has all the signs positive, and the remainder is $\frac{a^{m-1}}{2} \times$ hyp. log. $(x^2 - a^2)$.

'FLUENT 9.

(209.) To find the fluent of $\frac{x^{rm-1}\dot{x}}{x^m - a^m}$, where r is a whole positive number.

Divide the numerator by the denominator; the quotient is $x^{rm-m-1}\dot{x} + a^m x^{rm-2m-1}\dot{x} + a^{2m}x^{rm-3m-1}\dot{x} +$ &c. Let the division be continued for some number of terms as p; then the last term is $a^{m \cdot (p-1)}x^{rm-pm-1}\dot{x}$, and the remainder is $\frac{a^{pm}x^{rm-pm-1}\dot{x}}{x^m - a^m}$; if the variable part of this remainder were $\frac{x^{m-1}\dot{x}}{x^m - a^m}$, the fluent could be found, and it is evident that the division may be continued till it is. Assume, therefore, $rm - pm - 1 = m - 1$; then $p = r - 1$, or the number of terms in the quotient is $r - 1$. Hence, the last term is $a^{m \cdot (r-2)}x^{m-1}\dot{x}$, and the remainder is $\frac{a^{m \cdot (r-1)}x^{m-1}\dot{x}}{x^m - a^m}$;

therefore the required fluent $= \frac{x^{rm-m}}{rm - m} + \frac{a^m x^{rm-2m}}{rm - 2m} + \frac{a^{2m}x^{rm-3m}}{rm - 3m} +$ &c., $+ \frac{a^{m \cdot (r-2)}x^m}{m} +$ the remainder $\frac{a^{rm-m}}{m} \times$ hyp. log. $(x^m - a^m)$.

2. The fluent of $\frac{x^{rm-1}\dot{x}}{a^m + x^m}$ is found in the same manner.

In this case, the terms are alternately positive and negative; in other respects the fluents are the same.

3. The fluent of $\dfrac{x^{rm-1}\dot{x}}{a^m - x^m}$ has all its terms negative.

4. The fluent of $\dfrac{x^{rm-1}\dot{x}}{a \sim b\,x^m}$, or, of $\dfrac{1}{b} \times \dfrac{x^{rm-1}\dot{x}}{\frac{a}{b} \sim x^m}$ is the

same with the preceding, $\dfrac{a}{b}$ being substituted in the result

for a^m, and the whole series multiplied by the fraction $\dfrac{1}{b}$.

FLUENT 10.

(210.) **1.** To find the fluent of $\dfrac{c - ry^2}{c + by^2} \times y\dot{y}$.

This quantity $= \dfrac{cy\dot{y} - ry^3\dot{y}}{c + by^2}$. Divide, in an inverted

order,

$$by^2 + c\Big) - ry^3\dot{y} + cy\dot{y} \ \Big(\dfrac{-ryy}{b}$$

$$- ry^3\dot{y} - \dfrac{rcy\dot{y}}{b}$$

$$\overline{\left(\dfrac{bc + rc}{b}\right) \times \dfrac{y\dot{y}}{by^2 + c}}.$$

Hence the required fluent $= f. \dfrac{-ry\dot{y}}{b} + \dfrac{bc + rc}{b}$

$\times f. \dfrac{y\dot{y}}{by^2 + c} = \dfrac{-ry^2}{2b} + \dfrac{bc + rc}{2b^2} \times$ hyp. log. $\left(y^2 + \dfrac{c}{b}\right)$.

2. If the fluxion be $\dfrac{ar^2 - ry^2}{ar^2 + (a - r).y^2} \times y\dot{y}$, $c = ar^2$, and

$b = a - r$; therefore, the fluent is $\dfrac{-ry^2}{2 \times (a - r)} + \dfrac{\frac{1}{2}a^2r^2}{(a - r)^2} \times$

hyp. log. $\left(y^2 + \dfrac{ar^2}{a - r}\right)$.

3. The fluent of $\dfrac{ar^2+ry^2}{ar^2+(a+r).y^2} \times y\dot{y} = \dfrac{ry^2}{2\,(a+r)} + \dfrac{\frac{1}{2}\,a^2r^2}{(a+r)^2}$

$\times$ hyp. log. $\left(y^2 + \dfrac{ar^2}{a+r}\right)$.

SECTION II.

RULE. Many fluents may be found by dividing the given fluxion into two parts, and considering it as a rectangle. For, since the fluxion of $xy = x\dot{y} + y\dot{x}$, the fluent of $x\dot{y} = xy - f.\,y\dot{x}$. In these cases, the variable quantity without the vinculum, is divisible by the fluxion of that under the vinculum.

FLUENT 11.

(211.) To find the fluent of $- x^3\dot{x}.(a^2 - x^2)^{\frac{1}{2}}$.

Let $(a^2-x^2)^{\frac{1}{2}} \times -x\dot{x} = \dot{y}$, and $x^2 = z$; then the given flux-

ion $= z\dot{y}$; and its fluent $= zy - f.y\dot{z} = \dfrac{x^2}{3}\,(a^2-x^2)^{\frac{3}{2}} - \dfrac{2}{3} \times$

$f.x\dot{x}.(a^2-x^2)^{\frac{3}{2}} = \dfrac{x^2}{3}\,.(a^2-x^2)^{\frac{3}{2}} + \dfrac{2}{15}.(a^2-x^2)^{\frac{5}{2}} = \dfrac{x^2.(a^2-x^2)^{\frac{3}{2}}}{3}$

$+ \dfrac{2\,a^2 - 2x^2}{15}\,.(a^2-x^2)^{\frac{3}{2}} = \dfrac{3\,x^2+2\,a^2}{15}\,.(a^2-x^2)^{\frac{3}{2}}$.

FLUENT 12.

(212.) Required the fluent of $\dfrac{x\dot{x}}{(a+x)^{\frac{1}{2}}}$.

This $= x\dot{x}.(a+x)^{-\frac{1}{2}}$. Let $(a+x)^{-\frac{1}{2}} \times \dot{x} = \dot{y}$; then $x\dot{y} =$ the proposed fluxion; and the fluent $= xy - f.\,y\dot{x} =$

$$(a+x)^{\frac{1}{2}} \times 2x - f.(a+x)^{\frac{1}{2}} \times 2\dot{x} = (a+x)^{\frac{1}{2}} \times 2x - \frac{4}{3}(a+x)^{\frac{3}{2}}$$

$$= (a+x)^{\frac{1}{2}}.\left(2x - \frac{4a+4x}{3}\right) = \frac{(a+x)^{\frac{1}{2}}.(2x-4a)}{3}.$$

Fluent 13.

(213.) 1. Required the fluent of $\dfrac{bx^{2n-1}\dot{x}}{(a+fx^n)^{\frac{1}{2}}}$.

Assume $(a+fx^n)^{-\frac{1}{2}} \times x^{n-1}\dot{x} = \dot{y}$, and $bx^n = z$. Then the

fluent of $z\dot{y} = zy - f.y\dot{z} = bx^n \times \dfrac{2}{nf}.(a+fx^n)^{\frac{1}{2}} - \dfrac{2b}{nf} \times$

$f.(a+fx^n)^{\frac{1}{2}} \times nx^{n-1}\dot{x} = \dfrac{2b}{nf}.(a+fx^n)^{\frac{1}{2}} \times x^n - \dfrac{4b}{3nf^2}.$

$(a+fx^n)^{\frac{3}{2}}.$

By the same Rule,

2. The fluent of $\dfrac{-x^3\dot{x}}{(a^2-x^2)^{\frac{1}{2}}} = (a^2-x^2)^{\frac{1}{2}} \times \dfrac{2a^2+x^2}{3}$;

here assume $(a^2-x^2)^{-\frac{1}{2}} \times -x\dot{x} = \dot{y}$, and $x^2 = z$.

3. The fluent of $\dfrac{-x^5\dot{x}}{(a^2-x^2)^{\frac{1}{2}}} = \dfrac{x^4+4a^2x^2-8a^4}{3 \times (a^2-x^2)^{\frac{1}{2}}}.$

Here assume $(a^2-x^2)^{-\frac{1}{2}} \times -x\dot{x} = \dot{y}$, and $x^4 = z$.

4. The fluent of $(a+cz^n)^m \times dz^{2n-1}\dot{z} = \dfrac{d.(a+cz^n)^{m+1}}{nc^2}.$

$\left(\dfrac{cz^n}{m+2} - \dfrac{a}{(m+1).(m+2)}\right)$. Assume $(a+cz^n)^m \times z^{n-1}\dot{z}$

$= \dot{y}$, and $dz^n = x$.

SECTION III.

ON COMPARISON OF FLUENTS.

This is the method of deducing one fluent from another. It is not easy to lay down rules which will answer in every case where this method may be applied; but the general principle may be explained in the following manner.

RULE. Assume some quantity in the form of a rectangle $= p$; and of such a nature, that one part of its fluxion may be the fluxion of some known fluent, (as A); and the other part the fluxion $(\dot{N})$, whose fluent is required. Put this equal to $\dot{p}$; then since $\dot{N} = \pm \dot{p} \pm \dot{A}$, N itself $= \pm p \pm A$.

The most general method of making this assumption, and which obtains in many of the following fluents, is this. Divide the quantity without the vinculum, by the fluxion of the quantity under the vinculum, omit the constant quantities, and put this quotient $= p$, having first increased the index of the whole quantity under the vinculum by 1; and proceed in the manner stated above. In other cases the method will best appear by the Examples.

FLUENT 14.

(214.) Given the fluent of $\dfrac{\dot{x}}{(a^2+x^2)^{\frac{1}{2}}}$ (Art. 43.); required the fluent of $\dfrac{x^2\dot{x}}{(a^2+x^2)^{\frac{1}{2}}}$.

Assume $x \times (a^2 + x^2)^{\frac{1}{2}} = p$; $\therefore \dot{x} \times (a^2 + x^2)^{\frac{1}{2}} + \dfrac{x \dot{x} \times x}{(a^2 + x^2)^{\frac{1}{2}}} = \dot{p}$;

or, multiplying the numerator and denominator of the first part by $(a^2 + x^2)^{\frac{1}{2}}$, we have $\dfrac{a^2 \dot{x} + x^2 \dot{x} + x^2 \dot{x}}{(a^2 + x^2)^{\frac{1}{2}}} = \dot{p}$; $\therefore$

$\dfrac{x^2 \dot{x}}{(a^2 + x^2)^{\frac{1}{2}}} = \dfrac{\dot{p}}{2} - \dfrac{a^2 \dot{x}}{2 \times (a^2 + x^2)^{\frac{1}{2}}}$; and the fluent $= \dfrac{p}{2} -$

$f \cdot \dfrac{a^2 \dot{x}}{2 (a^2 + x^2)^{\frac{1}{2}}} = \dfrac{x \times (a^2 + x^2)^{\frac{1}{2}}}{2} - \dfrac{1}{2} a^2 \times$ hyp. log.

$\left(x + (a^2 + x^2)^{\frac{1}{2}} \right)$.

FLUENT 15.

(215.) 1. Given the fluent of $\dfrac{\dot{x}}{(a^2 - x^2)^{\frac{1}{2}}}$ (Art. 44.);

required the fluent of $\dfrac{x^2 \dot{x}}{(a^2 - x^2)^{\frac{1}{2}}}$.

Assume $x \times (a^2 - x^2)^{\frac{1}{2}} = p$; then, by taking the fluxion of this rectangle and reducing it, $\dfrac{x^2 \dot{x}}{(a^2 - x^2)^{\frac{1}{2}}} = \dfrac{a^2 \dot{x}}{2 (a^2 - x^2)^{\frac{1}{2}}}$

$- \dfrac{\dot{p}}{2}$; and the required fluent $= \dfrac{1}{2} a \times$ a circular arc of

radius a, and sine $x - \dfrac{x \times (a^2 - x^2)^{\frac{1}{2}}}{2}$.

This fluent is used in finding the area of the conchoid of Nicomedes.

2. By a similar process, the fluent of $\dfrac{x^2 \dot{x}}{(x^2 - a^2)^{\frac{1}{2}}} =$

$x \cdot \dfrac{(x^2 - a^2)^{\frac{1}{2}}}{2} + \dfrac{1}{2} a^2 \times$ hyp. log. $(x + \sqrt{x^2 - a^2})$.

FLUENT 16.

(216.) 1. Given the fluent of $\dfrac{x^3\dot{x}}{(a^2+x^2)^{\frac{1}{2}}}$ (A); to find the

$f.\ \dfrac{x^4\dot{x}}{(a^2+x^2)^{\frac{1}{2}}}$.

Assume $x^3\times(a^2+x^2)^{\frac{1}{2}}=p$; then, by taking the fluxion

and reducing the expression, we have $\dfrac{x^4\dot{x}}{(a^2+x^2)^{\frac{1}{2}}}=\dfrac{\dot{p}}{4}-$

$\dfrac{3a^2\times\dot{A}}{4}$; ∴ the required fluent $=x^3\cdot\dfrac{(a^2+x^2)^{\frac{1}{2}}}{4}-\dfrac{3a^2}{4}\times A.$

2. In the same manner, if the fluent of $\dfrac{x^3\dot{x}}{(a^2-x^2)^{\frac{1}{2}}}(B)$

be given, the fluent of $\dfrac{x^4\dot{x}}{(a^2-x^2)^{\frac{1}{2}}}=\dfrac{3a^2}{4}\times B-\dfrac{x^3\times(a^2-x^2)^{\frac{1}{2}}}{4}$.

3. If the fluent of $\dfrac{x^3\dot{x}}{(x^2-a^2)^{\frac{1}{2}}}$ (C) be given, the fluent

of $\dfrac{x^4\dot{x}}{(x^2-a^2)^{\frac{1}{2}}}=\dfrac{x^3\times(x^2-a^2)^{\frac{1}{2}}}{4}+\dfrac{3a^2}{4}\times C.$

4. And it is evident, that the fluents of $\dfrac{x^6\dot{x}}{(x^2\pm a^2)^{\frac{1}{2}}}$

$\dfrac{x^8\dot{x}}{(x^2\pm a^2)^{\frac{1}{2}}},\ \dfrac{x^n\dot{x}}{(x^2\pm a^2)^{\frac{1}{2}}}$, where n is an even number, may be

determined by knowing the fluent of the fluxion, which immediately precedes it in that series; and substituting

p as before. Thus, if the fluent of $\dfrac{x^{n-2}\dot{x}}{\sqrt{a^2+x^2}}=B$ be given,

the $f.\ \dfrac{x^n\dot{x}}{(a^2+x^2)^{\frac{1}{2}}}=\dfrac{x^{n-1}\cdot(a^2+x^2)^{\frac{1}{2}}}{n}-\dfrac{(n-1).a^2}{n}\times B.$

FLUENT 17.

(217.) 1. Given the fluent of $x\dot{x}\cdot(a^2+x^2)^{\frac{1}{2}}$ $(\dot{D})$; to find the fluent of $\dfrac{x^3\dot{x}}{(a^2+x^2)^{\frac{1}{2}}}$, where the index of x is an odd number.

Assume $x^2\cdot(a^2+x^2)^{\frac{1}{2}}=p$; then $2x\dot{x}\cdot(a^2+x^2)^{\frac{1}{2}}+\dfrac{x^3\dot{x}}{(a^2+x^2)^{\frac{1}{2}}}$
$=\dot{p}$. In this instance it is not necessary to reduce the first part, as in the former cases; for we have at once $\dfrac{x^3\dot{x}}{(a^2+x^2)^{\frac{1}{2}}}=\dot{p}-2\dot{D}$; $\therefore$ the required fluent
$=p-2D=x^2\times(a^2+x^2)^{\frac{1}{2}}-2D$, where $D=\dfrac{(a^2+x^2)^{\frac{3}{2}}}{3}$.

2. In the same manner, the fluent of $\dfrac{x^3\dot{x}}{(a^2-x^2)^{\frac{1}{2}}}=2D-$
$x^2\cdot(a^2-x^2)^{\frac{1}{2}}$, where $D=f.x\dot{x}.(a^2-x^2)^{\frac{1}{2}}=-\dfrac{(a^2-x^2)^{\frac{3}{2}}}{3}$.

This fluent is used in finding the content of the solid generated by the revolution of the conchoid about its axis.

3. And the fluent of $\dfrac{x^3\dot{x}}{(x^2-a)^{\frac{1}{2}}}=x^2.(x^2-a^2)^{\frac{1}{2}}-2D$, where
$D=f.x\dot{x}(x^2-a^2)^{\frac{1}{2}}=\dfrac{(x^2-a^2)^{\frac{3}{2}}}{3}$.

FLUENT 18.

(218.) 1. Given the fluent of $\dfrac{x^3\dot{x}}{(a^2+x^2)^{\frac{1}{2}}}$ $(\dot{E})$; to find
$f.\dfrac{x^5\dot{x}}{(a^2+x^2)^{\frac{1}{2}}}$.

Assume $x^4 \cdot (a^2 + x^2)^{\frac{1}{2}} = p$; then, by taking the fluxion and reducing it, $\dfrac{x^5 \dot{x}}{(a^2 + x^2)^{\frac{1}{2}}} = \dfrac{\dot{p}}{5} - \dfrac{4a^2 \dot{E}}{5}$; and the required

fluent $= \dfrac{x^4 \cdot (a^2 + x^2)^{\frac{1}{2}}}{5} - \dfrac{4a^2}{5} \times E.$

2. Thus also, the fluent of $\dfrac{x^5 \dot{x}}{(a^2 - x^2)^{\frac{1}{2}}}$, and of $\dfrac{x^5 \dot{x}}{(x^2 - a^2)^{\frac{1}{2}}}$, may be found.

3. In the same manner, the fluent of $\dfrac{x^7 \dot{x}}{(a^2 \pm x^2)^{\frac{1}{2}}}$, $\dfrac{x^9 \dot{x}}{(a^2 \pm x^2)^{\frac{1}{2}}}$, &c. $\dfrac{x^n \dot{x}}{(a^2 \pm x^2)^{\frac{1}{2}}}$ may be determined, where n is an odd number, by knowing the fluent of the fluxion, which immediately precedes it in that series. Thus, if the fluent of $\dfrac{x^{n-2}}{(a^2 + x^2)^{\frac{1}{2}}} = \dot{E}$ be given, the fluent of

$\dfrac{x^n \dot{x}}{(a^2 + x^2)^{\frac{1}{2}}} = \dfrac{x^{n-1} \cdot (a^2 + x^2)^{\frac{1}{2}}}{n} - \dfrac{(n-1) \cdot a^2}{n} \times E.$

FLUENT 19.

(219.) 1. Given the fluent of $\dfrac{\dot{x}}{(a+x)^{\frac{1}{2}}}$ ($\dot{F}$); to find the fluent of $\dfrac{x \dot{x}}{(a+x)^{\frac{1}{2}}}$.

Let $x \cdot (a + x)^{\frac{1}{2}} = p$; then $\dfrac{a \dot{x} + \frac{3}{2} x \dot{x}}{(a+x)^{\frac{1}{2}}} = \dot{p}$; and

$\dfrac{x \dot{x}}{(a+x)^{\frac{1}{2}}} = \dfrac{2\dot{p}}{3} - \dfrac{\frac{2}{3} a \dot{x}}{(a+x)^{\frac{1}{2}}}$; and the fluent $= \dfrac{2p}{3} - \dfrac{2a}{3} \times F$

$= \dfrac{2x \cdot (a+x)^{\frac{1}{2}}}{3} - \dfrac{2a}{3} \times F.$

2. In the same manner we can find the flu

$\dfrac{x\dot{x}}{(a-x)^{\frac{1}{4}}}$, and the fluent of $\dfrac{x\dot{x}}{(x-a)^{\frac{1}{4}}}$.

FLUENT 20.

(220.) 1. Given the fluent of $\dfrac{x\dot{x}}{(a+x)^{\frac{1}{2}}}$ $(\dot{G})$; required

the fluent of $\dfrac{x^2\dot{x}}{(a+x)^{\frac{1}{2}}}$.

Assume $x^2 . (a + x)^{\frac{1}{2}} = p$; then, taking the fluxion

and reducing it, $\dfrac{x^2\dot{x}}{(a+x)^{\frac{1}{2}}} = \dfrac{2\dot{p}}{5} - \dfrac{4ax\dot{x}}{5 . (a + x)^{\frac{1}{2}}}$; and the

fluent $= \dfrac{2x^2}{5} . (a+x)^{\frac{1}{2}} - \dfrac{4a \times G}{5}$.

2. In the same manner, if the fluent of $\dfrac{x^{n-1}\dot{x}}{(a+x)^{\frac{1}{2}}}$ $(\dot{G})$

be given, the fluent of $\dfrac{x^n\dot{x}}{(a+x)^{\frac{1}{2}}} = \dfrac{2}{2n+1} \times x^n \times (a+x)^{\frac{1}{2}} -$
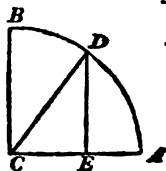
$\dfrac{2na}{2n+1} \times G$.

3. Thus also, the fluent of $\dfrac{x^n\dot{x}}{(a \sim x)^{\frac{1}{2}}}$ may be found in

terms of p, and the fluent, which precedes it, in the

series.

4. The fluent of $\dfrac{x\dot{x}}{(x - a)^{\frac{1}{2}}}$, by this method, $=$

$\dfrac{2x . (x - a)^{\frac{1}{2}}}{3} + \dfrac{4a . (x - a)^{\frac{1}{2}}}{3}$.

Lemma.

t, and C the center of the circle, d versed sine x, the $(2ax - x^2)^{\frac{1}{2}}$. Hence, ι area $AED = \dot{x} \times$ the area $AED =$ the fluent $\ldots \qquad /-x^2)^{\frac{1}{2}}$.

Fluent 21.

(221.) Given the fluent of $\dot{x} \cdot (2ax - x^2)^{\frac{1}{2}}$ $(\dot{H})$; required the fluent of $x\dot{x} \cdot (2ax - x^2)^{\frac{1}{2}}$.

Assume $(2ax - x^2)^{\frac{3}{2}} = p$; then $\frac{3}{2} \cdot (2a\dot{x} - 2x\dot{x}) \cdot (2ax - x^2)^{\frac{1}{2}} = \dot{p}$; or, $3a\dot{x} \cdot (2ax - x^2)^{\frac{1}{2}} - 3x\dot{x} \cdot (2ax - x^2)^{\frac{1}{2}} = \dot{p}$; $\therefore x\dot{x} \cdot (2ax - x^2)^{\frac{1}{2}} = a \times \dot{H} - \frac{\dot{p}}{3}$; and the required fluent $= a \times H - \dfrac{(2ax - x^2)^{\frac{3}{2}}}{3}$.

Fluent 22.

(222.) 1. Given the fluent of $x\dot{x} \cdot (2ax - x^2)^{\frac{1}{2}}$ $(\dot{I})$; required the fluent of $x^2\dot{x} \cdot (2ax - x^2)^{\frac{1}{2}}$.

Assume $x \cdot (2ax - x^2)^{\frac{3}{2}} = p$; then, by taking the fluxion and reducing it as before, $5a\dot{I} - 4x^2\dot{x} \cdot (2ax - x^2)^{\frac{1}{2}} = \dot{p}$; therefore the fluent of $x^2\dot{x} \cdot (2ax - x^2)^{\frac{1}{2}} = \dfrac{5a}{4} \times I - \dfrac{x \cdot (2ax - x^2)^{\frac{3}{2}}}{4}$.

2. In the same manner the series may be continued; and if the fluent of $x^{n-1}\dot{x} \cdot (2ax - x^2)^{\frac{1}{2}}$ $(\dot{I})$ be given, the fluent of $x^n\dot{x} \cdot (2ax - x^2)^{\frac{1}{2}} = \dfrac{(2n+1) \cdot aI - x^{n-1}(2ax - x^2)^{\frac{3}{2}}}{n+2}$.

Fluent 23.

223.) 1. Given the fluent of $\dfrac{\dot{x}}{(2ax+x^2)^{\frac{1}{2}}}$ ($\dot{K}$ Art. 43.);

iired the fluent of $\dfrac{x\dot{x}}{(2ax+x^2)^{\frac{1}{2}}}$.

ssume $(2ax+x^2)^{\frac{1}{2}} = p$; then $\dfrac{a\dot{x}+x\dot{x}}{(2ax+x^2)^{\frac{1}{2}}} = \dot{p}$;

$\therefore \dfrac{x\dot{x}}{(2ax+x^2)^{\frac{1}{2}}} = \dot{p} - a\dot{K}$; and the required fluent $=$

$(2ax+x^2)^{\frac{1}{2}} - a \times K.$

2. Thus also, if the fluent of $\dfrac{x\dot{x}}{(2ax+x^2)^{\frac{1}{2}}}$ be given, the

fluent of $\dfrac{x^2\dot{x}}{(2ax+x^2)^{\frac{1}{2}}}$ may be found, by assuming

$x \cdot (2ax+x^2)^{\frac{1}{2}} = p$, and so on, to any fluxion of this form.
If the fluent of $x^{n-1}\dot{x} \cdot (2ax+x^2)^{\frac{1}{2}}$ ($\dot{K}$) be given, by
assuming $x^{n-1} \cdot (2ax+x^2)^{\frac{1}{2}} = p$, we have the fluent of

$$\dfrac{x^n\dot{x}}{(2ax+x^2)^{\frac{1}{2}}} = \dfrac{x^{n-1} \cdot (2ax+x^2)^{\frac{1}{2}}}{n} - \dfrac{(2n-1).a}{n} \times K.$$

3. In the same way, the fluent of $\dfrac{\dot{x}}{(x^2-2ax)^{\frac{1}{2}}}$ (Art. 43.)

being given, the fluent of $\dfrac{x\dot{x}}{(x^2-2ax)^{\frac{1}{2}}}$ may be found;

and the series may be continued to $\dfrac{x^n\dot{x}}{(x^2-2ax)^{\frac{1}{2}}}$.

4. If the fluent of $\dfrac{x^{n-1}\dot{x}}{(x^2-2ax)^{\frac{1}{2}}}$ ($\dot{K}$) be given, by

assuming $x^{n-1} \cdot (x^2 - 2ax)^{\frac{1}{2}} = p$, the fluent of $\dfrac{x^n \dot{x}}{(x^2 - 2ax)^{\frac{1}{2}}} =$

$$\dfrac{x^{n-1} \cdot (x^2 - 2ax)^{\frac{1}{2}}}{n} + \dfrac{(2n-1) \cdot a}{n} \times K.$$

5. The fluent of $\dfrac{\dot{x}}{(2ax - x^2)^{\frac{1}{2}}}$ being given (Art. 44.),

that of $\dfrac{x\dot{x}}{(2ax - x^2)^{\frac{1}{2}}}$, $\dfrac{x^2 \dot{x}}{(2ax - x^2)^{\frac{1}{2}}}$, &c., $\dfrac{x^n \dot{x}}{(2ax - x^2)^{\frac{1}{2}}}$ may

be obtained by the same method of continuation.

The fluent of $\dfrac{x\dot{x}}{(ax - x^2)^{\frac{1}{4}}}$ found by this method $=$

a circular arc of radius $\frac{1}{2} a$, and versed sine x, $- (ax - x^2)^{\frac{1}{4}}$.

6. If the fluent of $\dfrac{x^{n-1}\dot{x}}{(2ax - x^2)^{\frac{1}{4}}}$ ($\dot{K}$) be given, and

$x^{n-1} \cdot (2ax - x^2)^{\frac{1}{4}} = p$, the fluent of $\dfrac{x^n \dot{x}}{(2ax - x^2)^{\frac{1}{4}}} = \dfrac{(2n-1)}{n}$

$\times aK - \dfrac{x^{n-1} \cdot (2ax - x^2)^{\frac{1}{4}}}{n}.$

7. The fluent of $\dfrac{x^2 \dot{x}}{(ax - x^2)^{\frac{1}{4}}}$ by this method $= \dfrac{3a}{4} \times$

a circular arc of radius $\frac{1}{2} a$, and versed sine x —

$\dfrac{x \cdot (ax - x^2)^{\frac{1}{4}}}{2} - \dfrac{3a}{4} \cdot (ax - x^2)^{\frac{1}{4}}$. This fluent is used in

finding the area of the cissoid.

8. Thus also the fluent of $\dot{x} \cdot (x - a)^{\frac{1}{4}}$ being given, the

fluent of $x\dot{x} \cdot (x - a)^{\frac{1}{4}}$, $x^2 \dot{x} \cdot (x - a)^{\frac{1}{4}}$, &c., $x^n \dot{x} \cdot (x - a)^{\frac{1}{4}}$

can be found.

If the fluent of $x^{n-1} \dot{x} . (x-a)^{\frac{1}{2}}$ $(\dot{K})$ be given, by assuming $x^n (x-a)^{\frac{3}{2}} = p$, the fluent of $x^n \dot{x} . (x-a)^{\frac{1}{2}}$ is found

$$= \frac{2 x^n . (x-a)^{\frac{3}{2}}}{2n+3} + \frac{2 na}{2n+3} \times K.$$

9. And, given the fluxion of $\dot{x} . (x-a)^m$, the fluent of $x \dot{x} . (x-a)^m$, the fluent of $x^2 \dot{x} . (x-a)^m$, &c., $x^n \dot{x} . (x-a)^m$ may be found.

10. If the fluent of $x^{n-1} \dot{x} . (x-a)^m$ $(\dot{K})$ be given, by assuming $x^n . (x-a)^{m+1} = p$, the fluent of $x^n \dot{x} (x-a)^m$ will be found $= \dfrac{x^n . (x-a)^{m+1} + naK}{m+n+1}$,

11. By the same mode of continuation, the fluent of $x^2 \dot{x} . (a \pm x)^{\frac{1}{2}}$, and of $x^2 \dot{x} . (a \pm x)^m$ may be found.

LEMMA.

If $GP = x$, and $GE = r$, $EF =$
$2(r^2 - x^2)^{\frac{1}{2}}$; and the fluxion of the
area $ABFE = EF \times$ the fluxion of
$GP = 2\dot{x} . (r^2 - x^2)^{\frac{1}{2}}$.

Cor. The fluent of $2\dot{x} . (r^2 - x^2)^{\frac{1}{2}}$
$=$ the area $ABFE$.
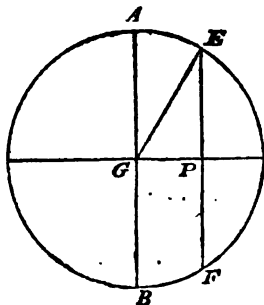
FLUENT 24.

(224.) Given the fluent of $\dot{x} . (r^2 - x^2)^{\frac{1}{2}}$ $(\dot{A},)$ to find the fluent of $x^2 \dot{x} . (r^2 - x^2)^{\frac{1}{2}}$.

Assume $x.(r^2 - x^2)^{\frac{3}{2}} = p$; then $\dot{x}.(r^2 - x^2)^{\frac{3}{2}} - 3x^2 \dot{x}.(r^2 - x^2)^{\frac{1}{2}}$

$=\dot{p}$; that is, $(r^3\dot{x}-4x^2\dot{x}).(r^3-x^2)^{\frac{1}{2}}=\dot{p}$; $\therefore x^2\dot{x}.(r^3-x^2)^{\frac{1}{2}}$

$=\dfrac{r^2\dot{x}}{4}.(r^3-x^2)^{\frac{1}{2}}-\dfrac{\dot{p}}{4}$; and the fluent $=\dfrac{r^2A-p}{4}$.

When $x=0$, $p=0$; $\therefore$ no correction is wanted. Also, when $x=r$, $p=0$; $\therefore$ the whole fluent between the values

of $x=0$ and $x=r$, is $\dfrac{r^2A}{4}=\dfrac{r^2}{4}\times\frac{1}{2}$ the semi-circle $BCA=$

$\dfrac{r^2}{16}\times$ the area of the whole circle $=\dfrac{r^2}{16}\times pr^2$, where $p=$

.3.14159, &c.

COR. 1. Hence, between similar values of x, the fluent

of $4x^2\dot{x}.(r^2-x^2)^{\frac{1}{2}}=\dfrac{r^2}{4}\times pr^2$.

COR. 2. Between the same values of x, the fluent of

$4a^2\dot{x}.(r^2-x^2)^{\frac{1}{2}}+4x^2\dot{x}.(r^2-x^2)^{\frac{1}{2}}=2a^2\times$ the area of

the semi-circle $ACB,+\dfrac{r^2}{4}\times pr^2=\left(a^2+\dfrac{r^2}{4}\right)\times pr^2$.

This fluent occurs in Ex. 9. Chap. xi. where it is re-
quired to find the center of oscillation of a circle.

FLUENT 25.

(225.) Given the fluent of $x^2\dot{x}.(r^2-x^2)^{\frac{1}{2}}$ $(\dot{B})$; to find

the fluent of $x^4\dot{x}.(r^2-x^2)^{\frac{1}{2}}$ $(\dot{C})$.

Assume $x^3.(r^2-x^2)^{\frac{3}{2}}=p$; then $3x^2\dot{x}.(r^2-x^2).(r^2-x^2)^{\frac{1}{2}}$

$-3x^4\dot{x}.(r^2-x^2)^{\frac{1}{2}}=\dot{p}$, or $3r^2\dot{B}-6\dot{C}=\dot{p}$; $\therefore C=\dfrac{3r^2B-p}{6}$

$=\dfrac{3r^2B-x^3.(r^2-x^2)^{\frac{3}{2}}}{6}$.

In the same manner, if the fluent of $x^{n-2}\dot{x}.(r^2-x^2)^{\frac{1}{2}}$ $(\dot{D})$

be given, where n is even, the fluent of $x^n \dot{x} . (r^2 - x^2)^{\frac{1}{2}}$

can be found by assuming $x^{n-1} (r^2 - x^2)^{\frac{3}{2}} = p$; the fluent

$$= \frac{(n-1) . r^2 D - p}{n+2} .$$

Cor. Hence the fluent of $\dot{x} . (r^2 - x^2)^{\frac{3}{2}}$, or the fluent

of $r^2 \dot{x} . (r^2 - x^2)^{\frac{1}{2}} - f . x^2 \dot{x} . (r^2 - x^2)^{\frac{1}{2}} = r^2 \times$ area $AGPE$

$- \dfrac{r^2}{4} \times$ area $AGPE + \dfrac{x . (r^2 - x^2)^{\frac{3}{2}}}{4} = \dfrac{3 r^2}{4} \times$ area $AGPE$

$+ \dfrac{x . (r^2 - x^2)^{\frac{3}{2}}}{4} .$

This fluent is used in Cotes' Problems, to compare the
momentum of a sphere and circumscribing cylinder re-
volving round a common axis.

FLUENT 26.

(226.) 1. Given the fluent of $x \dot{x} . (a^2 + x^2)^{\frac{1}{2}}$ (L); to

find the fluent of $x^3 \dot{x} . (a^2 + x^2)^{\frac{1}{2}}$.

Assume $x^2 . (a^2 + x^2)^{\frac{3}{2}} = p$; $\therefore 2 x \dot{x} . (a^2 + x^2) . (a^2 + x^2)^{\frac{1}{2}}$

$+ 3 x^3 \dot{x} . (a^2 + x^2)^{\frac{1}{2}} = \dot{p}$; $\therefore 2 a^2 \dot{L} + 2 x^3 \dot{x} . (a^2 + x^2)^{\frac{1}{2}} +$

$3 x^3 \dot{x} . (a^2 + x^2)^{\frac{1}{2}} = \dot{p}$; $\therefore x^3 \dot{x} . (a^2 + x^2)^{\frac{1}{2}} = \dfrac{\dot{p} - 2 a^2 \dot{L}}{5}$; and

the required fluent $= \dfrac{x^2 . (a^2 + x^2)^{\frac{3}{2}} - 2 a^2 L}{5} .$

2. In the same manner the fluent of $x^5 \dot{x} . (a^2 \pm x^2)^{\frac{1}{2}}$,

$x^7 \dot{x} . (a^2 \pm x^2)^{\frac{1}{2}}$ &c. may be found by means of the fluent
which precedes it in that series.

R

FLUENT 27.

(227.) 1. In general, given the fluent of $x^{n-1}\dot{x}.(a^2+x^2)^{\frac{1}{4}}$ $(\dot{M})$; to find the fluent of $x^n\dot{x}.(a^2+x^2)^{\frac{1}{4}}$, where n is an odd number.

Assume $x^{n-1}.(a^2+x^2)^{\frac{3}{4}}=p$, then $(n-1).x^{n-2}\dot{x}.(a^2+x^2)$ $(a^2+x^2)^{\frac{1}{4}}+3x^n\dot{x}.(a^2+x^2)^{\frac{1}{4}}=\dot{p}$, or $(n-1).a^2\dot{M}+(n+2)$ $x^n\dot{x}(a^2+x^2)^{\frac{1}{4}}=\dot{p}$, and the fluent required $=$

$$\frac{x^{n-1}.(a^2+x^2)^{\frac{3}{4}}-(n-1).a^2M}{n+2}.$$

2. The fluent of $x^n\dot{x}.(x^2\pm a^2)^{\frac{1}{4}}$ may be found from similar data.

FLUENT 28.

(228.) Given the fluent of $(a+cz^n)^m\times dz^{n-1}\dot{z}$ $(\dot{L})$; to find the fluent of $(a+cz^n)^m\times dz^{2n-1}\dot{z}$ $(\dot{M})$.

Assume $(a+cz^n)^{m+1}\times dz^n=p$; then $(m+1).ncz^{n-1}\dot{z}.$ $(a+cz^n)^m\times dz^n+ndz^{n-1}\dot{z}.(a+cz^n)^{m+1}=\dot{p}$, or $(m+1)$ $ncdz^{2n-1}\dot{z}.(a+cz^n)^m+ndaz^{n-1}\dot{z}.(a+cz^n)^m+ndcz^{2n-1}\dot{z}$ $(a+cz^n)^n=\dot{p}$; that is, by collecting the quantities $(m+2).nc\dot{M}+na\dot{L}=\dot{p}$; $\therefore M=\dfrac{p-naL}{(m+2).nc}$; or the

required fluent $=\dfrac{(a+cz^n)^{m+1}\times dz^n}{(m+2).nc}-\dfrac{a}{(m+2).c}.$

$\dfrac{(a+cz^n)^{m+1}\times d}{(m+1).nc}.$

FLUENT 29.

(229.) Given the fluent of $(a+cz^n)^m \times dz^{2n-1} \dot{z} \ (\dot{M})$; required the fluent of $(a+cz^n)^m \times dz^{3n-1} \dot{z} (\dot{N})$.

Assume $(a+cz^n)^{m+1} \times dz^{2n} = p$; then by taking the fluxion and reducing it, as in the last case, we get $(m+3)$ $nc \times \dot{N} = \dot{p} - 2an\dot{M}$; $\therefore$ N, or the fluent required, $=$

$$\frac{p-2anM}{(m+3).nc} = \frac{(a+cz^n)^{m+1} \times dz^{2n}}{(m+3).nc} - \frac{2a}{(m+3).c} \times M.$$

In the same manner the series of fluents may be continued to $(a+cz^n)^m \times dz^{rn-1} \dot{z}$, where r is a whole number.

For another method of finding this fluent, see Ex. 74, under Transformations.

FLUENT 30.

(230.) Given the fluent of $(a+cz^n)^m \times z^{rn-1} \dot{z} \ (\dot{P})$; to find the fluent of $(a+cz^n)^m \times z^{rn+n-1} \dot{z} \ (\dot{Q})$.

Assume $(a+cz^n)^{m+1} \times z^{rn} = p$; $\therefore$ $(m+1).nc\, z^{n-1}\, \dot{z} \times$ $(a+cz^n)^m \times z^{rn} + nr.(a+cz^n).(a+cz^n)^m \times z^{rn-1} \dot{z} = \dot{p}$; that is, $(m+1).nc.\dot{Q} + nar\dot{P} + nrc\dot{Q} = \dot{p}$; $\therefore$ $(m+r+1)$ $nc\dot{Q} = \dot{p} - nar\dot{P}$; $\therefore$ $Q = \dfrac{(a+cz^n)^{m+1} \times z^{rn} - narP}{(m+r+1).nc}$.

FLUENT 31.

(231.) Given the fluent of $\dot{P}$ as before; to find the fluent of $(a+cz^n)^m \times z^{rn-n-1} \dot{z} \ (\dot{R})$.

Assume $(a+cz^n)^{m+1} \times z^{rn-n} = p$; $\therefore$ $(m+1).ncz^{n-1}\dot{z}$. $(a+cz^n)^m \times z^{rn-n} + (zn-n).z^{rn-n-1}\dot{z}.(a+cz^n).(a+cz^n)^m$

$=\dot{p}$; that is, $(m+1).nc\dot{P} + (rn-n).a\dot{R} + (rn - n)\times$ $c\dot{P} = \dot{p}$; or, $(m + r).nc\dot{P} + (r-1).na\dot{R}=\dot{p}$; $\therefore R=$

$$\frac{p - (m+r).ncP}{(r-1).na} = \frac{(a+cz^n)^{m+1} \times z^{rn-n} - (m+r).ncP}{(r-1).na}.$$

FLUENT 32.

(232.) Given the fluent of $(a+cz^n)^m \times z^{rn+n-1} \dot{z}$ $(\dot{S})$; to find the fluent of $(a+cz^n)^{m+1} \times z^{rn-1} \dot{z}$ $(\dot{T})$.

Assume $(a+cz^n)^{m+1} \times z^{rn}=p$, then $(m+1).ncz^{n-1}\dot{z}.$ $(a+cz^n)^m . z^{rn} + nrz^{rn-1}\dot{z}.(a+cz^n)^{m+1}=\dot{p}$; or, $(m+1)$ $\times nc\dot{S} + nr\dot{T} = \dot{p}$; $\therefore T = \dfrac{p - (m+1).ncS}{nr} =$

$$\frac{(a+cz^n)^{m+1} \times z^{rn} - (m+1).ncS}{nr}.$$

FLUENT 33.

(233.) Given the fluent of $(a+cz^n)^m.z^{rn-n-1}\dot{z}$ $(\dot{V})$; to find the fluent of $(a+cz^n)^{m-1} \times z^{rn-1} \dot{z}$ $(\dot{W})$.

Assume $(a+cz^n)^m \times z^{rn-n}=p$; $\therefore mcnz^{n-1}\dot{z}.(a+cz^n)^{m-1}$ $z^{rn-n} + (rn - n).z^{rn-n-1}\dot{z}.(a+cz^n)^m=\dot{p}$; or, $mcn\dot{W}+$ $(rn - n) \times \dot{V} = \dot{p}$; $\therefore W = \dfrac{p - (rn - n).V}{mcn} =$

$$\frac{(a+cz^n)^m . z^{rn-n} - (rn-n).V}{mcn}.$$

FLUENT 34.

(234.) Given the fluent of $(a+cz^n)^{m+1} \times z^{rn-1} \dot{z}(\dot{X})$; to find the fluent of $(a+cz^n)^{m+1} \times z^{rn+n-1} \dot{z}$ $(\dot{Y})$.

Assume $(a+cz^n)^{m+2} \times z^{rn}=p$; $\therefore (m + 2).ncz^{n-1}\dot{z} .$ $(a+cz^n)^{m+1} \times z^{rn} + rnz^{rn-1}\dot{z}.(a+cz^n).(a+cz^n)^{m+1}=\dot{p}$;

or, $(m+2).nc\dot{Y} + rna\dot{X} + rnc\dot{Y} = \dot{p}$; $\therefore (m+r+2)$

$\times nc\dot{Y} = \dot{p} - rna\dot{X}$; $\therefore Y = \dfrac{p - rnaX}{(m+r+2).nc} =$

$\dfrac{(a+cz^n)^{m+2} \times z^m - rnaX}{(m+r+2).nc}$.

FLUENT 35.

(235.) Given the fluent of $(a+cz^n)^{m-1} \times z^{rn-1}\dot{z}$ $(\dot{Z})$; to find the fluent of $(a+cz^n)^{m-1} \times z^{rn-s-1}\dot{z}$ $(\dot{A})$.

Assume $(a+cz^n)^m \times z^{rn-s} = p$; $\therefore mncz^{s-1}\dot{z}.(a+cz^n)^{m-s}$

$\times z^{rn-s} + (rn-n).z^{rn-s-1}\dot{z}.(a+cz^n).(a+cz^n)^{m-1}$

$= \dot{p}$; or, $mnc\dot{Z} + (rn-n).a\dot{A} + (rn-n).c\dot{Z} = \dot{p}$; $\therefore A =$

$\dfrac{p-(m+r-1).ncZ}{(rn-n).a} = \dfrac{(a+cz^n)^m \times z^{rn-s} - (m+r-1).ncZ}{(r-1).na}$.

FLUENT 36.

(236.) Given the fluent of $(a+cz^n)^m \times z^r\dot{z}$ $(\dot{B})$; to find the fluent of $(a+cz^n)^m \times z^{r+n}\dot{z}$ $(\dot{C})$.

Assume $(a+cz^n)^{m+1} \times z^{r+1} = p$; then taking the fluxions,

$(m+1).nc.\dot{C} + (r+1).a\dot{B} + (r+1).c\dot{C} = \dot{p}$;

$\therefore (mn+n+r+1).c\dot{C} = \dot{p} - (r+1).a\dot{B}$; $\therefore C =$

$\dfrac{p-(r+1).aB}{(mn+n+r+1).c} = \dfrac{(a+cz^n)^{m+1} \times z^{r+1} - (r+1).aB}{(mn+n+r+1).c}$.

FLUENT 37.

(237.) Given the fluent of $(a+cz^n)^m . z^{r+n}\dot{z}$ $(\dot{C})$; to find the fluent of $(a+cz^n)^{m+1} \times z^r\dot{z}$ $(\dot{D})$.

Assume $(a+cz^n)^{m+1} \times z^{r+1} = p$; then $(m+1).nc\dot{C} +$

$(r+1).\dot{D} = \dot{p}$; $\therefore D = \dfrac{(a+cz^n)^{m+1} \times z^{r+1} - (m+1).ncC}{r+1}$.

FLUENT 38.

(238.) Given the fluent of $(a + cz^n)^m \times z^r \dot{z}$ $(\dot{A})$; to find the fluent of $(a + cz^n)^m \times z^{r+n}\dot{z}$ $(\dot{B})$, and of $(a + cz^n)^{m+1} \times z^r\dot{z}$ $(\dot{C})$.

Assume $(a + cz^n)^{m+1} \times z^{r+1} = p$; then $(m+1) . ncz^{r+n}\dot{z} .$ $(a + cz^n)^m + (r+1) . z^r\dot{z} . (a + cz^n)^{m+1} = \dot{p}$; or, $(m+1)$ $\times nc\dot{B} + (r+1).\dot{C} = \dot{p}$; $\therefore B = \dfrac{p - (r+1).C}{(m+1) . nc}$.

Also, $\dot{C} = z^r\dot{z} . (a + cz^n)^{m+1} = az^r\dot{z} . (a + cz^n)^m + cz^{r+n}\dot{z} .$ $(a + cz^n)^m = a\dot{A} + c\dot{B}$; $\therefore B = \dfrac{C - aA}{c}$. Hence $\dfrac{p - (r+1).C}{(m+1) . nc} = \dfrac{C - aA}{c}$; $\therefore p - (r+1). C = (m+1).nC -$ $(m+1).naA$; $\therefore C = \dfrac{p + (m+1) . naA}{r + 1 + mn + n}$. Whence $B =$ $\dfrac{C - aA}{c} = \dfrac{p + (m+1) . naA}{(r + 1 + mn + n) . c} - \dfrac{aA}{c}$.

Cor. By increasing m each time by 1, and r by n, the fluent may be continued as far as we please.

SECTION IV.

ON THE TRANSFORMATION OF FLUXIONS.

Rule I. Many fluxions which are not of the common form, in Art. 39., may be reduced to that form by a transposition of the variable quantity, and the fluents found by that Rule.

FLUENT 39.

(239.) To find the fluent of $\dfrac{\dot{x}}{x^2 \cdot (a^2 - x^2)^{\frac{1}{2}}}$.

Since $a^2 - x^2 = x^2 \cdot (a^2 x^{-2} - 1)$; $\therefore (a^2 - x^2)^{\frac{1}{2}} = x \times$

$(a^2 x^{-2} - 1)^{\frac{1}{2}}$; hence $\dfrac{\dot{x}}{x^2 \cdot (a^2 - x^2)^{\frac{1}{2}}} = \dfrac{x^{-3}\dot{x}}{(a^2 x^{-2} - 1)^{\frac{1}{2}}} =$

$x^{-3}\dot{x} \cdot (a^2 x^{-2} - 1)^{-\frac{1}{2}}$; and the fluent $= \dfrac{2 \cdot (a^2 x^{-2} - 1)^{\frac{1}{2}}}{-2a^2} =$

$-\dfrac{(a^2 - x^2)^{\frac{1}{2}}}{a^2 x}$.

In the same manner the fluent of $\dfrac{\dot{x}}{x^2 \cdot (a^2 + x^2)^{\frac{1}{2}}} = -$

$\dfrac{(a^2 + x^2)^{\frac{1}{2}}}{a^2 x}$.

FLUENT 40.

(240.) To find the fluent of $\dfrac{a\dot{x}}{(a^2 + x^2)^{\frac{3}{2}}}$.

Since $a^2 + x^2 = x^2 \cdot (a^2 x^{-2} + 1)$; $\therefore \dfrac{a\dot{x}}{(a^2 + x^2)^{\frac{3}{2}}} =$

$a x^{-3}\dot{x} \cdot (a^2 x^{-2} + 1)^{-\frac{3}{2}}$; and the fluent $= \dfrac{(a^2 x^{-2} + 1)^{-\frac{1}{2}}}{a} =$

$\dfrac{x}{a \cdot (a^2 + x^2)^{\frac{1}{2}}}$.

This fluent is used in Prob. 1., Art. 138., on the Attraction of Bodies.

The fluent of $\dfrac{a\dot{x}}{(a^2 - x^2)^{\frac{3}{2}}} = \dfrac{x}{a \cdot (a^2 - x^2)^{\frac{1}{2}}}$.

And by the same process, if $a = 1$, we have the fluents of

$$\frac{\dot{x}}{(1+x^2)^{\frac{1}{2}}} = \frac{x}{(1+x^2)^{\frac{1}{2}}}, \text{ and of } \frac{\dot{x}}{(1-x^2)^{\frac{1}{2}}} = \frac{x}{(1-x^2)^{\frac{1}{2}}}.$$

FLUENT 41.

(241.) To find the fluent of $\dfrac{x^{-\frac{1}{2}n-1}\dot{x}}{(a+cx^n)^{\frac{1}{2}}}$.

Since $a + cx^n = x^n . (ax^{-n} + c)$; $\therefore (a + cx^n)^{\frac{1}{2}} = x^{\frac{n}{2}}.$
$(ax^{-n}+c)^{\frac{1}{2}}$, and the given fluxion $= x^{-n-1}\dot{x} . (ax^{-n}+c)^{-\frac{1}{2}}$;

and the fluent $= -\dfrac{2}{na} \times \dfrac{(a+cx^n)^{\frac{1}{2}}}{x^{\frac{n}{2}}}$.

The fluent of $\dfrac{x^{-\frac{1}{2}n-1}\dot{x}}{(a-cx^n)^{\frac{1}{2}}} = -\dfrac{2}{na} \times \dfrac{(a-cx^n)^{\frac{1}{2}}}{x^{\frac{n}{2}}}$.

FLUENT 42.

(242.) To find the fluent of $\dfrac{\dot{x} . (a^2+x^2)^{\frac{1}{2}}}{x^4}$.

Since $a^2 + x^2 = x^2 . (a^2x^{-2} + 1)$; $\therefore \dfrac{\dot{x} . (a^2 + x^2)^{\frac{1}{2}}}{x^4} =$

$x^{-3}\dot{x} . (a^2x^{-2} + 1)^{\frac{1}{2}}$; and the fluent $= -\dfrac{1}{3a^2x^3} . (a^2 + x^2)^{\frac{3}{2}}$.

Rule II. If the index of the variable quantity without the vinculum increased by 1, be some aliquot part of the corresponding index under the vinculum, substitute for that power of the variable quantity, which is obtained by dividing the index under the vinculum by the number which expresses the aliquot part.

FLUENT 43.

(243.) To find the fluent of $\dfrac{x\dot{x}}{(a^4-x^4)^{\frac{1}{2}}}$.

Let $y = x^2$, and $a^4 = b^2$; then the given fluxion $=$ $\dfrac{\frac{1}{2}\dot{y}}{(b^2 - y^2)^{\frac{1}{2}}}$, and the fluent $= \dfrac{1}{2b} \times$ circular arc of radius b, and sine y.

2. In the same manner the fluent of $\dfrac{x\dot{x}}{(a^4 + x^4)^{\frac{1}{2}}} = \frac{1}{2}$ hyp. log. $\left(y + (b^2 + y^2)^{\frac{1}{2}}\right)$.

3. And the fluent of $\dfrac{x\dot{x}}{(x^4 - a^4)^{\frac{1}{2}}} = \frac{1}{2}$ hyp. log. $\left(y + (y^2 - b^2)^{\frac{1}{2}}\right)$.

FLUENT 44.

(244.) To nd the fluent of $\dfrac{z^{\frac{1}{2}n-1}\dot{z}}{(c^n \pm z^n)^{\frac{1}{2}}}$.

1. Assume $c^{\frac{n}{2}} = a$; and $z^{\frac{n}{2}} = x$; then $z^{\frac{1}{2}n-1}\dot{z} = \dfrac{2\dot{x}}{n}$, and the fluxion $\dfrac{z^{\frac{1}{2}n-1}\dot{z}}{(c^n + z^n)^{\frac{1}{2}}}$ is transformed into $\dfrac{\frac{2}{n}\dot{x}}{(a^2 + x^2)^{\frac{1}{2}}}$; of which the fluent is $\dfrac{2}{n} \times$ hyp. log. $\left(x + (a^2 + x^2)^{\frac{1}{2}}\right)$.

2. The fluxion $\dfrac{z^{\frac{1}{2}n-1}\dot{z}}{(c^n - z^n)^{\frac{1}{2}}}$ may be transformed by the same process into $\dfrac{\frac{2}{n} \times \dot{x}}{(a^2 - x^2)^{\frac{1}{2}}} = \dfrac{2}{na} \times \dfrac{a\dot{x}}{(a^2 - x^2)^{\frac{1}{2}}}$; whose fluent $= \dfrac{2}{na} \times$ a circular arc, whose radius $= a$, and sine $= x$.

(245.) We have in these cases supposed the quantity in the denominator to be affected by a radical; the same method will apply where the radical is not found.

1. Thus the fluent of $\dfrac{x\dot{x}}{a^2-x^4} = \dfrac{1}{4b} \times$ hyp. log. $\dfrac{b+y}{b-y}$, where $y = x^2$, and $b = a^2$.

2. The fluent of $\dfrac{x\dot{x}}{a^4+x^4} = \dfrac{1}{2b^2} \times$ a circular arc of radius b, and tangent y.

3. The fluent of $\dfrac{x\dot{x}}{x^4-a^4} = \dfrac{1}{4b} \times$ hyp. log. $\dfrac{y-b}{y+b}$.

FLUENT 45.

(246.)　1. To find the fluent of $\dfrac{x^{\frac{1}{2}n-1}\dot{x}}{c^n+x^n}$ ↓

Let $c^{\frac{n}{2}} = a$, $x^{\frac{n}{2}} = x$; then the given fluxion becomes $=$

$\dfrac{\frac{2}{n} \times \dot{x}}{a^2+x^2} = \dfrac{2}{na^2} \times \dfrac{a^2\dot{x}}{a^2+x^2}$; and the fluent $= \dfrac{2}{na^2} \times$ a circular arc of radius a, and tangent x.

2. To find the fluent of $\dfrac{x^{\frac{1}{2}n-1}\dot{x}}{c^n-x^n}$.

This may be transformed in the same manner into $\dfrac{2}{n} \times \dfrac{\dot{x}}{a^2-x^2}$, or $\dfrac{1}{na} \times \dfrac{2a\dot{x}}{a^2-x^2}$; whose fluent $= \dfrac{1}{na} \times$ hyp. log. $\dfrac{a+x}{a-x}$.

If the variable quantity under the radical have a coefficient, bring it without the radical.

FLUENT 46.

(247.) Required the fluent of $\dfrac{x^{\frac{1}{2}n-1}\dot{x}}{(a+cx^n)^{\frac{1}{2}}}$.

This $= \dfrac{z^{\frac{1}{2}n-1}\dot{z}}{c^{\frac{1}{2}}\left(\frac{a}{c}+z^n\right)^{\frac{1}{2}}}$. Let $\dfrac{a}{c} = b^2$ and $z^{\frac{n}{2}} = x$; then

the fluxion becomes $\dfrac{2}{nc^{\frac{1}{2}}} \times \dfrac{\dot{x}}{(b^2+x^2)^{\frac{1}{2}}}$; and the fluent

$= \dfrac{2}{nc^{\frac{1}{2}}} \times \text{hyp. log.} \left(x+(b^2+x^2)^{\frac{1}{2}}\right)$.

In the same manner the fluent of $\dfrac{z^{\frac{1}{2}n-1}\dot{z}}{(a-cz^n)^{\frac{1}{2}}} = \dfrac{2}{nbc^{\frac{1}{2}}} \times$

a circular arc of radius b, and sine x.

FLUENT 47.

(248.) 1. To find the fluent of $\dfrac{\dot{z}}{z.(a+cz^n)^{\frac{1}{2}}}$.

This $= \dfrac{\dot{z}}{z . c^{\frac{1}{2}} . \left(\frac{a}{c}+z^n\right)^{\frac{1}{2}}}$. Let $z^{\frac{n}{2}} = x$, then $z = x^{\frac{2}{n}}$

and $\dot{z} = \dfrac{2}{n} \times x^{\frac{2}{n}-1}\dot{x}$; $\therefore \dfrac{\dot{z}}{z} = \dfrac{2}{n} \times \dfrac{x^{\frac{2}{n}-1}\dot{x}}{x^{\frac{2}{n}}} = \dfrac{2\dot{x}}{nx}$; $\therefore \dfrac{\dot{z}}{c^{\frac{1}{2}}z.\left(\frac{a}{c}+z^n\right)^{\frac{1}{2}}}$

$= \dfrac{2\dot{x}}{nc^{\frac{1}{2}}x.\left(\frac{a}{c}+x^2\right)^{\frac{1}{2}}} = \left(\text{if } \dfrac{a}{c} = b^2\right) \dfrac{2\dot{x}}{nc^{\frac{1}{2}} \times x.(b^2+x^2)^{\frac{1}{2}}} =$

$\dfrac{1}{nbc^{\frac{1}{2}}} \times \dfrac{2b\dot{x}}{x.(b^2+x^2)^{\frac{1}{2}}}$; whose fluent $= \dfrac{1}{nbc^{\frac{1}{2}}} \times \text{hyp. log.}$

$\dfrac{(b^2+x^2)^{\frac{1}{2}} - b}{(b^2+x^2)^{\frac{1}{2}} + b} = \dfrac{1}{na^{\frac{1}{2}}} \times \text{hyp. log.} \dfrac{(a+cz^n)^{\frac{1}{2}} - a^{\frac{1}{2}}}{(a+cz^n)^{\frac{1}{2}} + a^{\frac{1}{2}}}$.

When a is negative the fluent fails, for $\dfrac{1}{na^{\frac{1}{2}}}$ is impossible.

2. By the same substitution the fluxion of $\dfrac{\dot{z}}{z.(cz^n - a)^{\frac{1}{2}}}$

may be reduced to $\dfrac{2}{n\,b^2\,c^{\frac{1}{2}}} \times \dfrac{b^2\,\dot{x}}{x.(x^2 - b^2)^{\frac{1}{2}}}$; whose fluent $=$

$\dfrac{2}{n\,b^2\,c^{\frac{1}{2}}} \times$ a circular arc of radius b, and secant x.

3. And the fluent of $\dfrac{\dot{z}}{z.(a - cz^n)^{\frac{1}{2}}} = \dfrac{1}{n\,b\,c^{\frac{1}{2}}} \times$ hyp.

$\log . \dfrac{b - (b^2 - x^2)^{\frac{1}{2}}}{b + (b^2 - x^2)^{\frac{1}{2}}}$.

FLUENT 48.

(249.) To find the fluent of $\dfrac{r\,x^{\frac{n-5}{2}}\,\dot{x}}{(r^{n-3} - x^{n-3})^{\frac{1}{2}}}$.

1. Let $x^{n-3} = z^2$, or $z = x^{\frac{n-3}{2}}$; then $\dot{z} = \dfrac{n-3}{2} \times x^{\frac{n-5}{2}}\dot{x}$;

$\therefore \dfrac{2\,\dot{z}}{n-3} = x^{\frac{n-5}{2}}\dot{x}$; hence the given flux^a. $= \dfrac{2\,r\,\dot{z}}{(n-3).(r^{n-3} - z^2)^{\frac{1}{2}}}$

$= \dfrac{-2r}{(n-3).r^{\frac{n-3}{2}}} \times \dfrac{-r^{\frac{n-3}{2}}\,\dot{z}}{(r^{n-3} - z^2)^{\frac{1}{2}}}$; and the fluent $=$

$\dfrac{-2r}{(n-3).r^{\frac{n-3}{2}}} \times$ a circular arc of radius $r^{\frac{n-3}{2}}$, and cosine z;

or $x^{\frac{n-3}{2}}$.

2. In the same manner the fluent of $\dfrac{-r\,x^{\frac{n-5}{2}}\,\dot{x}}{(r^{n-3} - x^{n-3})^{\frac{1}{2}}} =$

$\dfrac{+2r}{(n-3).r^{\frac{n-3}{2}}} \times$ a circular arc of radius $r^{\frac{n-3}{2}}$, and cosine $x^{\frac{n-3}{2}}$.

These fluents are used in Art. 163. and 164. Sect. 4. Chap. xix.

RULE III. When the power or powers of the variable quantity without the radical are mostly in the denominator, it may be useful to substitute for the reciprocal of the quantity.

FLUENT 49.

(250.) Required the fluent of $\dfrac{\dot{z}}{z^2.(a^2+z^2)^{\frac{1}{4}}}$.

Let $x=\dfrac{1}{z}$, then $-\dot{x}=\dfrac{\dot{z}}{z^2}$; and the given fluxion $=$

$\dfrac{-\dot{x}}{\left(a^2+\dfrac{1}{x^2}\right)^{\frac{1}{2}}}=\dfrac{-x\dot{x}}{(a^2x^2+1)^{\frac{1}{4}}}=\left(\text{if }b=\dfrac{1}{a}\right)\dfrac{-bx\dot{x}}{(x^2+b^2)^{\frac{1}{4}}}$; whose

fluent $=-b.(x^2+b^2)^{\frac{1}{2}}=-\dfrac{1}{a}\cdot\left(\dfrac{1}{z^2}+\dfrac{1}{a^2}\right)^{\frac{1}{2}}=-\dfrac{(a^2+z^2)^{\frac{1}{2}}}{a^2z}$.

FLUENT 50.

(251.) To find the fluent of $\dfrac{\dot{z}}{z.(z^2+az)^{\frac{1}{4}}}$.

Let $z=\dfrac{1}{x}$; then, $x=\dfrac{1}{z}$ and $-\dot{x}\times z=\dfrac{\dot{z}}{z}$, or $-\dfrac{\dot{x}}{x}=\dfrac{\dot{z}}{z}$;

and the given fluxion $=\dfrac{-\dot{x}}{x.\left(\dfrac{1}{x^2}+\dfrac{a}{x}\right)^{\frac{1}{2}}}=\dfrac{-x\dot{x}}{x.(1+ax)^{\frac{1}{4}}}=$

$\left(\text{if }b=\dfrac{1}{a}\right)\dfrac{-b^{\frac{1}{2}}\dot{x}}{(b+x)^{\frac{1}{2}}}$; and the fluent $=-2b^{\frac{1}{4}}.(b+x.)^{\frac{1}{4}}$

FLUENT 51.

(252.) By a similar substitution, and the application of the Rule in Sect. 2. Art. 211., the fluents of the following fluxions may be found.

1. The fluent of $\dfrac{\dot{x}}{x^4.(a^2+x^2)^{\frac{1}{4}}} = (a^2+x^2)^{\frac{1}{4}}.\dfrac{(2x^2-a^2)}{3a^4 \times x^3}$.

2. The fluent of $\dfrac{\dot{x}.(a^2+x^2)^{\frac{1}{3}}}{x^6} = \dfrac{(2x^2-3a^2)}{15a^4} \times \dfrac{(a^2+x^2)^{\frac{3}{4}}}{x^5}$.

3. The fluent of $\dfrac{\dot{x}}{x^2.(a^2+x^2)^{\frac{3}{4}}} = \dfrac{-a^2+2x^2}{a^4x.(a^2+x^2)^{\frac{1}{4}}}$.

4. The fluent of $\dfrac{\dot{x}}{x^2.(a^2+x^2)^{\frac{5}{4}}}$, &c. &c.

FLUENT 52.

(253.) To find the fluent of $\dfrac{\dot{x}}{(a+x)^2.(a^2-x^2)^{\frac{1}{4}}}$.

Let $\dfrac{1}{a+x}=y$; $\therefore -\dot{y}=\dfrac{\dot{x}}{(a+x)^2}$; and $a+x=\dfrac{1}{y}$; $\therefore x=$

$\dfrac{1}{y}-a$, and $x^2=\dfrac{1}{y^2}-\dfrac{2a}{y}+a^2$; $\therefore (a^2-x^2)^{\frac{1}{4}}=\left(\dfrac{2a}{y}-\dfrac{1}{y^2}\right)^{\frac{1}{4}}=$

$\dfrac{(2ay-1)^{\frac{1}{4}}}{y}$; $\therefore$ the flux$^n.=\dfrac{-y\dot{y}}{(2ay-1)^{\frac{1}{4}}}=\dfrac{-y\dot{y}}{\overline{2a|}^{\frac{1}{4}}\left(y-\dfrac{1}{2a}\right)^{\frac{1}{4}}}$,

which is of the same form with Fluent 19.

FLUENT 53.

(254.) To find the fluent of $\dfrac{z\dot{z}}{(a+z)^3.(a^2+az+z^2)^{\frac{1}{2}}}$.

Let $x=\dfrac{a^2}{a+z}$; then $z=\dfrac{a^2-ax}{x}=a \times \dfrac{a-x}{x}$; $\therefore \dot{z}=$

$-\dfrac{a^2\dot{x}}{x^2}$; $z\dot{z}=-\dfrac{a^3\dot{x}.(a-x)}{x^3}$; $(a^2+az+z^2)^{\frac{1}{2}}=\dfrac{a}{x} \times$

$(a^2 - a x + x^2)^{\frac{1}{2}}$; $\therefore$ the given fluxion $= \dfrac{x^2 \dot{x} - a x \dot{x}}{a^4 . (a^2 - ax + x^2)^{\frac{1}{2}}}$,

of which the fluent is found in Fluents 82, 83.

In some cases it is useful to substitute not for the reciprocal of the denominator, but for the denominator itself.

FLUENT 54.

(255.) To find the fluent of $\dfrac{L x \dot{x} + M \dot{x}}{(x-p)^2}$.

Let $x - p = z$; $x = z + p$, and $\dot{x} = \dot{z}$; hence the given

fluxion $= \dfrac{Lz \dot{z} + Lp\dot{z} + M \dot{z}}{z^2} = $ (if $Lp + M = b$) $\dfrac{Lz\dot{z} + b\dot{z}}{z^2} =$

$\dfrac{L\dot{z}}{z} + \dfrac{b\dot{z}}{z^2}$; whose fluent is $L \times$ hyp. log. $z - \dfrac{b}{z} = L \times$

hyp. log. $(x - p) - \dfrac{b}{x - p}$.

FLUENT 55.

(256.) To find the fluent of $\dfrac{a z \dot{z} - 3az^3 \dot{z}}{(1 + z^2)^3}$.

Let $1 + z^2 = x$; then $z^2 = x - 1$, and $z^4 = x^2 - 2x + 1$;

$\therefore z^3 \dot{z} = \dfrac{x \dot{x}}{2} - \dfrac{\dot{x}}{2}$; also $z\dot{z} = \dfrac{\dot{x}}{2}$; and $(1 + z^2)^3 = x^3$; $\therefore$ the

given fluxion $= \dfrac{\dfrac{1}{2}a\dot{x} - \dfrac{3a}{2} \times x\dot{x} + \dfrac{3a}{2} \times \dot{x}}{x^3} = \dfrac{2 a \dot{x}}{x^3} - \dfrac{3 a \dot{x}}{2 x^2}$;

and the fluent $= \dfrac{3a}{2x} - \dfrac{a}{x^2} = \dfrac{3a}{2 . (1 + z^2)} - \dfrac{a}{(1 + z^2)^2}$.

RULE IV. The given fluxion may frequently be reduced to a better form, by actual multiplication or division.

FLUENT 56.

(257.)   The fluent of $\dfrac{x^{\frac{1}{2}}\dot{x}}{(a+x)^{\frac{1}{2}}} = f.\dfrac{x\,\dot{x}}{(ax+x^{\nu})^{\frac{1}{2}}}$, $x^{\frac{1}{2}}$ being

multiplied both into the numerator and denominator;

and the fluent (Fluent 23.) $= (ax+x^{2})^{\frac{1}{2}} - \dfrac{1}{2}a \times$ hyp.

log. $\left(x + \dfrac{a}{2} + (x^{2}+ax)^{\frac{1}{2}}\right).$

FLUENT 57.

(258.) The fluent of $\dfrac{x^{\frac{3}{4}}\dot{x}}{(a+x)^{\frac{1}{2}}} = f.\dfrac{x^{2}\dot{x}}{(ax+x^{2})^{\frac{1}{2}}}$, which

corresponds with Fluent 23.

FLUENT 58.

(259.)   The fluent of $\dot{x}.(x^{2} \pm a^{2})^{\frac{1}{2}} =$ the fluent of

$\dfrac{x^{2}\dot{x} \pm a^{2}\dot{x}}{(x^{2} \pm a^{2})^{\frac{1}{2}}}$; the fluent of the first part is given in Fluent 14,

and of the last in Art. 43.

2.   The fluents of $x^{2}\dot{x}.(a^{2} \pm x^{2})^{\frac{1}{2}}$ &c.; $x^{n}\dot{x}.(a^{2} \pm x^{2})^{\frac{1}{2}}$

where n is an even number may be found in the same

way.

3.   The fluent of $\dfrac{\dot{y}.(y^{2}+b^{2})^{\frac{1}{2}}}{b}$ by this method $= \dfrac{1}{b} \times$

$f.\dfrac{y^{2}\dot{y}}{(y^{2}+b^{2})^{\frac{1}{2}}} + \dfrac{b^{2}\dot{y}}{(y^{2}+b^{2})^{\frac{1}{2}}} = \dfrac{1}{b} \times \left(\dfrac{y.(b^{2}+y^{2})^{\frac{1}{2}}}{2} - \frac{1}{2}b^{2} \times \right.$ hyp.

log.$\left(y+(b^{2}+y^{2})^{\frac{1}{2}}\right) + b^{2} \times$ hyp. log.$\left.\left(y+(b^{2}+y^{2})^{\frac{1}{2}}\right)\right) = \dfrac{1}{2b}$

$(y^{4}+b^{2}y^{2})^{\frac{1}{2}} + \frac{1}{2}b \times$ hyp. log. $\left(y+(b^{2}+y^{2})^{\frac{1}{2}}\right).$ This fluent

is used in finding the length of the common parabola, Ex. 2. Art. 55; of the Spiral of Archimedes, Prob. 6. Art. 125; and of the Surface of a Solid generated by the revolution of the logarithmic curve about its axis, Art. 136. Prob. 6.

FLUENT 59.

(260.) The fluent of $\dfrac{\dot{x} \cdot (2ax + x^2)^{\frac{1}{2}}}{x} = f \cdot \dfrac{2a\dot{x}}{(2ax + x^2)^{\frac{1}{2}}} +$

$f \cdot \dfrac{x\dot{x}}{(2ax + x^2)^{\frac{1}{2}}}$; for which see Art. 43., and Fluent 23.

Thus also the fluents of $\dfrac{\dot{x} \cdot (2ax - x^2)^{\frac{1}{2}}}{x}$ and of

$\dfrac{\dot{x} \cdot (x^2 - 2ax)^{\frac{1}{2}}}{x}$ by Fluent 23.

FLUENT 60.

(261.) 1. The fluent of $\dfrac{\dot{x} \cdot (a^2 + x^2)^{\frac{1}{2}}}{x} = f \cdot \dfrac{a^2 \dot{x}}{x.(a^2 + x^2)^{\frac{1}{2}}} +$

$f \cdot \dfrac{x\dot{x}}{(a^2 + x^2)^{\frac{1}{2}}} = \dfrac{a}{2} \times \text{hyp. log.} \dfrac{(a^2 + x^2)^{\frac{1}{2}} - a}{(a^2 + x^2)^{\frac{1}{2}} + a} + (a^2 + x^2)^{\frac{1}{2}}.$

This fluent is used in finding the length of the logarithmic curve, Art. 136. Prob. 5.

2. The fluent of $\dfrac{\dot{x} \cdot (x^2 - a^2)^{\frac{1}{2}}}{x} = f \cdot \dfrac{x\dot{x}}{(x^2 - a^2)^{\frac{1}{2}}} - f \cdot \dfrac{a^2 \dot{x}}{x.(x^2 - a^2)^{\frac{1}{2}}}$

$= (x^2 - a^2)^{\frac{1}{2}} -$ a circular arc of radius a, and secant x.

FLUENT 61.

(262.) The fluent of $\dfrac{a\dot{x}}{x^{\frac{1}{2}} \cdot (a + x)^{\frac{1}{2}}} = f \cdot \dfrac{a\dot{x}}{(ax + x^2)^{\frac{1}{2}}} = a \times$

hyp. log. $\left(x + \dfrac{a}{2} + (x^2 + ax)^{\frac{1}{2}} \right).$

RULE V. The given fluxion may frequently be reduced to a better form, by the addition and subtraction of another quantity, or by dividing it into other parts.

FLUENT 62.

(263.) To find the fluent of $\dfrac{x\dot{x}}{a\pm x}$.

1. $\dfrac{x\dot{x}}{a+x} = \dot{x} + \dfrac{x\dot{x}}{a+x} - \dot{x} = \dot{x} - \dfrac{a\dot{x}}{a+x}$; $\therefore \int . \dfrac{x\dot{x}}{a+x} = x -$

$a \times$ hyp. log. $(a+x)$.

2. $\dfrac{x\dot{x}}{a-x} = \dot{x} + \dfrac{x\dot{x}}{a-x} - \dot{x} = \dfrac{a\dot{x}}{a-x} - \dot{x}$; $\therefore \int . \dfrac{x\dot{x}}{a-x} = -x - a \times$

hyp. log. $(a-x)$.

3. $\dfrac{x\dot{x}}{x-a} = \dot{x} + \dfrac{x\dot{x}}{x-a} - \dot{x} = \dot{x} + \dfrac{a\dot{x}}{x-a}$; $\therefore \int . \dfrac{x\dot{x}}{x-a} = x + a \times$

hyp log. $(x-a)$.

FLUENT 63.

(264.) To find the fluent of $\dfrac{x^2\dot{x}}{a^2 \pm x^2}$.

1. $\dfrac{x^2\dot{x}}{a^2+x^2} = \dot{x} + \dfrac{x^2\dot{x}}{a^2+x^2} - \dot{x} = \dot{x} - \dfrac{a^2\dot{x}}{a^2+x^2}$; and the fluent$=$

$x -$ a circular arc of radius a, and tangent x.

2. In the same manner, the fluent $\dfrac{x^2\dot{x}}{a^2-x^2} = \dfrac{a}{2} \times$ hyp.

log. $\dfrac{a+x}{a-x} - x$.

3. And the fluent $\dfrac{x^2\dot{x}}{x^2-a^2} = x + \dfrac{a}{2} \times$ hyp. log. $\dfrac{x-a}{x+a}$.

FLUENT 64.

(265.) To find the fluent of $\dfrac{x^3 \dot{x}}{a^2 \pm x^2}$.

1. $\dfrac{x^3 \dot{x}}{a^2 + x^2} = x\dot{x} + \dfrac{x^3 \dot{x}}{a^2 + x^2} - x\dot{x} = x\dot{x} - \dfrac{a^2 x \dot{x}}{a^2 + x^2}$; $\therefore \displaystyle\int . \dfrac{x^3 \dot{x}}{a^2 + x^2}$

$= \dfrac{x^2}{2} - \dfrac{a^2}{2} \times$ hyp. log. $(a^2 + x^2)$.

2. The $\displaystyle\int . \dfrac{x^3 \dot{x}}{a^2 - x^2}$ in the same manner $= -\dfrac{x^2}{2} - \dfrac{a^2}{2} \times$ hyp. log. $(a^2 - x^2)$.

3. And $\displaystyle\int . \dfrac{x^3 \dot{x}}{x^2 - a^2} = \dfrac{x^2}{2} + \dfrac{a^2}{2} \times$ hyp. log. $(x^2 - a^2)$.

FLUENT 65.

(266.) To find the fluent of $\dfrac{\dot{x}}{x} \times \dfrac{1}{1 - x}$.

1. This $= \dfrac{\dot{x}}{x - x^2} = \dfrac{\dot{x} - 2x\dot{x}}{x - x^2} + \dfrac{2x\dot{x}}{x - x^2} = \dfrac{\dot{x} - 2x\dot{x}}{x - x^2} + \dfrac{2\dot{x}}{1 - x}$;

and the fluent $=$ hyp. log. $(x - x^2) - 2 \times$ hyp. log. $(1 - x) =$ hyp. log. $(x - x^2) -$ hyp. log. $(1 - x)^2$.

2. In the same manner, the $\displaystyle\int . \dfrac{\dot{x}}{x} \times \dfrac{1}{1 + x} =$ hyp. log. $(x + x^2) -$ hyp. log. $(1 + x)^2$.

FLUENT 66.

(267.) To find the fluent of $\dfrac{\dot{x}.(2ax + x^2)^{\frac{1}{2}}}{x}$.

This $= \dfrac{2a\dot{x}}{(2ax + x^2)^{\frac{1}{2}}} + \dfrac{x\dot{x}}{(2ax + x^2)^{\frac{1}{2}}} = \dfrac{a\dot{x} + x\dot{x}}{(2ax + x^2)^{\frac{1}{2}}} +$

$\dfrac{a\dot{x}}{(2ax+x^2)^{\frac{1}{2}}}$; and the fluent $= (2ax+x^2)^{\frac{1}{2}} + a \times$ hyp. log.

of $\left(x + a + (2ax+x^2)^{\frac{1}{2}}\right)$ (Art. 43).

For another method, see Fluent 59.

FLUENT 67.

(268.) The fluent of $y^{-\frac{1}{2}}\dot{y}\cdot(a-y)^{\frac{1}{2}} = f.\dfrac{a\dot{y} - y\dot{y}}{(ay - y^2)^{\frac{1}{2}}} =$

$f.\dfrac{\frac{1}{2}a\dot{y}-y\dot{y}}{(ay-y^2)^{\frac{1}{2}}} + f.\dfrac{\frac{1}{2}a\dot{y}}{(ay-y^2)^{\frac{1}{2}}} = (ay-y^2)^{\frac{1}{2}} +$ a circular

arc of radius $\frac{1}{2}$ a, and versed sine y.

FLUENT 68.

(269.) The fluent of $\dot{y}\cdot\left(\dfrac{y}{a-y}\right)^{\frac{1}{2}} = f.\dfrac{y^{\frac{1}{2}}\dot{y}}{(a-y)^{\frac{1}{2}}} =$

$f.\dfrac{y\dot{y}}{(ay-y^2)^{\frac{1}{2}}} = -1 \times f.\dfrac{\frac{1}{2}a\dot{y} - y\dot{y}}{(ay-y^2)^{\frac{1}{2}}} + f.\dfrac{\frac{1}{2}a\dot{y}}{(ay-y^2)^{\frac{1}{2}}} = -$

$(ay-y^2)^{\frac{1}{2}} +$a circular arc of radius $\frac{1}{2}$ a, and versed sine y.

FLUENT 69.

(270.) The fluent of $\dfrac{x\dot{x}}{(ax+x^2)^{\frac{1}{2}}} = f.\dfrac{\frac{1}{2}a\dot{x}+x\dot{x}}{(ax+x^2)^{\frac{1}{2}}} -$

$f.\dfrac{\frac{1}{2}a\dot{x}}{(ax+x^2)^{\frac{1}{2}}} = (ax+x^2)^{\frac{1}{2}} - \frac{1}{2}a \times$ hyp. log. $\left(x+\dfrac{a}{2} +\right.$

$\left.(x^2 + ax)^{\frac{1}{2}}\right)$.

FLUENT 70.

(271.) The fluent of $\dfrac{x\dot{x}}{a^2 + bx} = f.\dfrac{1}{b} \times \dfrac{x\dot{x}}{c + x}$

$\left(\text{if } c = \dfrac{a^2}{b}\right) = \dfrac{1}{b}\times f.\dfrac{c\dot{x}+x\dot{x} - c\dot{x}}{c+x} = \dfrac{1}{b}\times\left(f.\dot{x} - f.\dfrac{c\dot{x}}{c+x}\right) =$

$$\frac{1}{b} \times \left(x - c \times \text{hyp. log. } (c+x)\right) = \frac{1}{b} \times x - \frac{a^2}{b^2} \times \text{hyp. log.}$$

$$\left(\frac{a^2}{b} + x\right).$$

FLUENT 71.

(272.) 1. The fluent of $\dfrac{\dot{x}}{(2x-x^2)^{\frac{1}{4}}} = f. \dfrac{\dot{x} - x\dot{x}}{(2x-x^2)^{\frac{1}{4}}} +$

$f. \dfrac{x\dot{x}}{(2x-x^2)^{\frac{3}{4}}} = f. (\dot{x} - x\dot{x}) \cdot (2x - x^2)^{-\frac{3}{4}} + x^{-1}\dot{x} \cdot$

$(2x^{-1} - 1)^{-\frac{3}{4}} = -(2x - x^2)^{-\frac{1}{4}} + \dfrac{1}{(2x^{-1} - 1)^{\frac{1}{4}}} =$

$\dfrac{-1}{(2x - x^2)^{\frac{1}{2}}} + \dfrac{x^{\frac{1}{2}}}{(2 - x)^{\frac{1}{2}}} = -\dfrac{1}{(2x - x^2)^{\frac{1}{2}}} + \dfrac{x}{(2x - x^2)^{\frac{1}{2}}} =$

$\dfrac{x-1}{(2x-x^2)^{\frac{1}{2}}}.$

2. In the same manner, the fluent of $\dfrac{\dot{x}}{(2x+x^2)^{\frac{3}{4}}} =$

$-\dfrac{1}{(2x+x^2)^{\frac{1}{4}}} - \dfrac{x}{(2x+x^2)^{\frac{1}{2}}} = -\dfrac{x+1}{(2x+x^2)^{\frac{1}{2}}}.$

3. The fluent of $\dfrac{x\dot{x}}{(x - \frac{1}{4}a)^{\frac{1}{2}}} = f. \dfrac{x^2\dot{x} - \frac{1}{6}ax\dot{x}}{(x^3 - \frac{1}{4}ax^2)^{\frac{1}{2}}} +$

$f. \dfrac{\frac{1}{6}a\dot{x}}{(x - \frac{1}{4}a)^{\frac{1}{2}}} = \dfrac{2}{3} \cdot (x^3 - \frac{1}{4}ax^2)^{\frac{1}{2}} + \dfrac{a}{3} \cdot (x - \frac{1}{4}a)^{\frac{1}{2}}.$

RULE VI. When the given fluxion is affected by two different surds, and the rational quantity without the vinculum is in a given ratio to the fluxion of the variable quantity under it, it may be useful to substitute for one of the surds.

FLUENT 72.

(273.) To find the fluent of $\dfrac{z\dot{z}\,.\,(a^2-z^2)^{\frac{1}{2}}}{(z^2-b^2)^{\frac{1}{2}}}$.

Assume $x=(z^2-b^2)^{\frac{1}{2}}$; $\therefore x^2=z^2-b^2$; and $x\dot{x}=z\dot{z}$. Also $-b^2-x^2=-z^2$; $\therefore a^2-b^2-x^2=a^2-z^2$, or if $c^2=a^2-b^2$, $(c^2-x^2)^{\frac{1}{2}}=(a^2-z^2)^{\frac{1}{2}}$; therefore the fluxion $=$

$\dfrac{x\dot{x}\,.\,(c^2-x^2)^{\frac{1}{2}}}{x}=\dot{x}\,.\,(c^2-x^2)^{\frac{1}{2}}=\dfrac{c^2\dot{x}-x^2\dot{x}}{(c^2-x^2)^{\frac{1}{2}}}$; whose fluent

is found in Art. 44. and Fluent 15. The same Rule may be applied in cases like the following.

FLUENT 73.

(274.) To find the fluent of $\dfrac{z\dot{z}\,.\,(a^2-z^2)^{\frac{1}{2}}}{b^2+z^2}$.

1. Let $b^2+z^2=x^2$; then $z\dot{z}=x\dot{x}$. Also $(a^2-z^2)^{\frac{1}{2}}=(a^2+b^2-x^2)^{\frac{1}{2}}=(c^2-x^2)^{\frac{1}{2}}$, if $c^2=a^2+b^2$; therefore the given fluxion $=\dfrac{x\dot{x}\,.\,(c^2-x^2)^{\frac{1}{2}}}{x^2}=\dfrac{\dot{x}\,.\,(c^2-x^2)^{\frac{1}{2}}}{x}$; whose

fluent $=\dfrac{c}{2}\times$ hyp. log. $\dfrac{c-(c^2-x^2)^{\frac{1}{2}}}{c+(c^2-x^2)^{\frac{1}{2}}}+(c^2-x^2)^{\frac{1}{2}}$.

2. The fluent of $\dfrac{z\dot{z}\,.\,(b^2+z^2)^{\frac{1}{2}}}{(c^2-z^2)^{\frac{1}{2}}}$ is found in the

same manner. If x be assumed $=(c^2-z^2)^{\frac{1}{2}}$, the fluent will be found $=-$ a circular area $BCED$ whose radius is $(b^2+c^2)^{\frac{1}{2}}$ and cosine x.
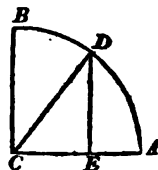
FLUENT 74.

(275.) Nearly in the same manner we may find the fluent of $(a+cz^n)^m\times dz^{n-1}\dot{z}$, where the index of z

without the vinculum, increased by unity, is some multiple of the index of x under the vinculum,

Let $a + cx^n = x$; then $x^n = \dfrac{x-a}{c}$; and $x^{rn} = \dfrac{(x-a)^r}{c^r}$;

$\therefore rnx^{rn-1}\dot{x} = \dfrac{r.(x-a)^{r-1}\dot{x}}{c^r}$; and $x^{rn-1}\dot{x} = \dfrac{1}{nc^r}.(x-a)^{r-1}\dot{x}$;

or if $r - 1 = s$, $dx^{rn-1}\dot{x} = \dfrac{d}{nc^r}.(x-a)^s \times \dot{x} =$ (by the

binomial theorem,) $\dfrac{d\dot{x}}{nc^r}.\left(x^s - sax^{s-1} + s.\dfrac{s-1}{2}a^2x^{s-2} - \&c.\right)$

Now for $dx^{rn-1}\dot{x}$ substitute this quantity, and x^m for $(a+cx^n)^m$, and the given fluxion will become $\dfrac{d}{nc^r} \times$

$x^m\dot{x}.\left(x^s - sax^{s-1} + s.\dfrac{s-1}{2}a^2x^{s-2} - \&c.\right) = \dfrac{d}{nc^r}.$

$\left(x^{m+s}\dot{x} - sax^{m+s-1}\dot{x} + s.\dfrac{s-1}{2}a^2x^{m+s-2}\dot{x} - \&c.\right)$ Hence the

fluent $= \dfrac{d}{nc^r}.\left(\dfrac{x^{m+s+1}}{m+s+1} - \dfrac{sax^{m+s}}{m+s} + \dfrac{s.\dfrac{s-1}{2}a^2x^{m+s-1}}{m+s-1}\right.$

$- \&c.\Big)$

On this Fluent it is useful to observe,

1. The series, arising from the expansion of $(x-a)^s$, will terminate, if r, and consequently s, be a whole positive number; therefore if m be a *positive* whole number, or a *positive* or *negative* fraction, the fluent can be found.

2. If r be a *positive* whole number, and m a *negative* whole number greater than $s + 1$, or r, the fluent can be found.

3. But if m be a *negative* whole number equal to or less than r, the variable part of one of the terms becomes $\dfrac{\dot{x}}{x}$,

which is found not by the common forms, but by
logarithms.

4. By Reduction, the original fluxion becomes
$(ax^{-n}+c)^m . dx^{(m+r)n-1}\dot{x}$; therefore if m and r be both
fractions, but $m+r$ a *negative* whole number, the fluent
can be found by transforming the fluxion as before. The
series will always terminate, and the fluent of each term
may be obtained by the common method.

FLUENT 75.

(276.) If $a=r^2, c=-1, n=2, m=\frac{1}{2}, d=1$, and $r=2$,
this fluxion becomes $x^3\dot{x}.(r^2-x^2)^{\frac{1}{2}}$. To find the fluent,
let $(r^2-x^2)^{\frac{1}{2}}=y$; then $x^2=r^2-y^2$, and $x^4=r^4-2r^2y^2+y^4$;
$\therefore x^3\dot{x}=-r^2y\dot{y}+y^3\dot{y}$; and $x^3\dot{x}.(r^2-x^2)^{\frac{1}{2}}=-r^2y^2\dot{y}+y^4\dot{y}$;
whose fluent $=-\frac{1}{3}r^2y^3+\frac{1}{5}y^5$. Now when $x=0$, sup-
pose $y=r$; the cor. $=\frac{2r^5}{15}$; and the whole corrected flu-
ent, when $x=r$ (or $y=0$), is $\frac{2r^5}{15}$. This fluent is used
in finding the center of gyration of a sphere. For
another method, see Fluent 11.

RULE VII. A trinomial may be reduced to a binomial,
by substituting for the variable quantity and half the
coefficient of the middle term with its proper sign.

FLUENT 76.

(277.) Required the fluent of $\dfrac{\dot{z}}{(b^2+cz+z^2)^{\frac{1}{2}}}$.

Let $x=z+\frac{1}{2}c$; then $x^2=z^2+cz+\frac{1}{4}c^2$; $\therefore x^2+b^2-\frac{c^2}{4}=$
z^2+cz+b^2; that is, if $a^2=b^2-\frac{c^2}{4}$, the given fluxion $=$
$\dfrac{\dot{x}}{(a^2+x^2)^{\frac{1}{2}}}$; and the fluent $=$ hyp. log. $\left(x+(a^2+x^2)^{\frac{1}{2}}\right)$.

FLUENT 77.

(278.) Required the fluent of $\dfrac{z^{n-1}\dot{z}}{(a+b\,z^n+cz^{2n})^{\frac{1}{2}}}$.

This $=\dfrac{z^{n-1}\dot{z}}{c^{\frac{1}{2}}.\left(\dfrac{a}{c}+\dfrac{bz^n}{c}+z^{2n}\right)^{\frac{1}{2}}}$. Assume $z^n+\dfrac{b}{2c}=x$;

then $z^{2n}+\dfrac{bz^n}{c}+\dfrac{b^2}{4c^2}=x^2$, and $\left(z^{2n}+\dfrac{bz^n}{c}+\dfrac{a}{c}\right)^{\frac{1}{2}}=$

$\left(x^2+\dfrac{a}{c}-\dfrac{b^2}{4c^2}\right)^{\frac{1}{2}}=(x^2\pm d^2)^{\frac{1}{2}}$; if $d^2=\dfrac{a}{c}-\dfrac{b^2}{4c^2}$, and $\dfrac{\dot{x}}{n}=$

$z^{n-1}\dot{z}$; therefore the given fluxion $=\dfrac{\dot{x}}{nc^{\frac{1}{2}}.(x^2\pm d^2)^{\frac{1}{2}}}$; whose

fluent $=\dfrac{1}{nc^{\frac{1}{2}}}\times$ hyp. log. $\left(x+(x^2\pm d^2)^{\frac{1}{2}}\right)$.

FLUENT 78.

(279.) Required the fluent of $\dot{z}.(a+bz+cz^2)^{\frac{1}{2}}$.

Assume $z+\dfrac{b}{2c}=x$; then by the last case the fluxion

may be reduced to the form $\dot{x}\,c^{\frac{1}{2}}\,(x^2+d^2)^{\frac{1}{2}}$, whose fluent
is found in Art. 259. Fluent 58.

FLUENT 79.

(280.) Required the fluent of $z^{n-1}\dot{z}.(a+bz^n+cz^{2n})^{\frac{1}{2}}$.

Assume $z^n+\dfrac{b}{2c}=x$; $\therefore z^{n-1}\dot{z}=\dfrac{\dot{x}}{n}$; and $z^{2n}+\dfrac{bz^n}{c}+$

$\dfrac{b^2}{4c^2}=x^2$; $\therefore \left(z^{2n}+\dfrac{bz^n}{c}+\dfrac{a}{c}\right)^{\frac{1}{2}}=(x^2\pm d^2)^{\frac{1}{2}}$, when $\pm d^2=$

$\dfrac{a}{c}-\dfrac{b^2}{4c^2}$; and the given fluxion $=\dfrac{\dot{x}\,c^{\frac{1}{2}}\,(x^2\pm d^2)^{\frac{1}{2}}}{n}$, whose

fluent is found in Fluent 58.

FLUENT 80.

(281.) Required the fluent of $\dfrac{z^{n-1}\,\dot{z}}{a+bz^{n}+cz^{2n}}$.

Assume $z^{n}+\dfrac{b}{2c}=x$; then the fluxion may be reduced

to the form $\dfrac{\dot{x}}{nc.(x^{2}\pm d^{2})}$, and the fluent found as before.

FLUENT 81.

(282.) To find the fluent of $\dfrac{cx\,\dot{x}-d\dot{x}}{x^{2}-ax+b}$.

Let $x-\dfrac{a}{2}=z$; then $\dot{x}=\dot{z}$, and $x^{2}-ax+\dfrac{a^{2}}{4}=z^{2}$; $\therefore$ $x^{2}-$

$ax+b=z^{2}+b-\dfrac{a^{2}}{4}=\left(\text{if }e^{2}=b-\dfrac{a^{2}}{4}\right)z^{2}\pm e^{2}$, according as

e^{2} is positive or negative; that is, according as the two values of x are impossible, or possible. Also $cx\,\dot{x}-$

$d\dot{x}=cz\dot{z}+\dfrac{ac\dot{z}}{2}-d\dot{z}$; therefore the given fluxion is

transformed into $\dfrac{cz\,\dot{z}+(\frac{1}{2}ac-d).\dot{z}}{z^{2}\pm e^{2}}=(\text{if }\frac{1}{2}ac-d=m)$

$\dfrac{cz\,\dot{z}+m\dot{z}}{z^{2}\pm e^{2}}$; and the fluent $=\frac{1}{2}c\times$ hyp. log. $(z^{2}\pm e^{2})+\dfrac{m}{e}\times$

a circular arc, of radius e and tangent z, if e^{2} be positive;

and if e^{2} be negative, the last part $=\dfrac{m\dot{z}}{z^{2}-e^{2}}=-\dfrac{m}{2e}\times$

$\dfrac{2e\dot{z}}{e^{2}-z^{2}}$, and its fluent is $-\dfrac{m}{2e}\times$ hyp. log. $\dfrac{e+z}{e-z}$. Let the

fluent of this second part $=A$; then the whole required fluent is $\frac{1}{2}c\times$ hyp. log. $(z^{2}\pm e^{2})+A$.

FLUENT 82.

(283.) To find the fluent of $\dfrac{x\dot{x}}{(a^2 - ax + x^2)^{\frac{1}{2}}}$.

Assume $x - \dfrac{a}{2} = y$; then, as before, $(x^2 - ax + a^2)^{\frac{1}{2}} = \left(y^2 + \dfrac{3a^2}{4}\right)^{\frac{1}{2}}$.

Also $x^2 = y^2 + ay + \dfrac{a^2}{4}$; $\therefore x\dot{x} = y\dot{y} + \frac{1}{2}a\dot{y}$; and the

given fluxion $= \dfrac{y\dot{y}}{\left(y^2 + \dfrac{3a^2}{4}\right)^{\frac{1}{2}}} + \dfrac{\frac{1}{2}a\dot{y}}{\left(y^2 + \dfrac{3a^2}{4}\right)^{\frac{1}{2}}}$; and the

fluent $= \left(y^2 + \dfrac{3a^2}{4}\right)^{\frac{1}{2}} + \frac{1}{2}a \times$ hyp. log. of $\left(y + \left(y^2 + \dfrac{3a^2}{4}\right)^{\frac{1}{2}}\right)$.

FLUENT 83.

(284.) In the same manner the fluent of $\dfrac{x^2\dot{x}}{(a^2 - ax + x^2)^{\frac{1}{2}}}$

may be resolved into $\dfrac{y^2\dot{y}}{\left(y^2 + \dfrac{3a^2}{4}\right)^{\frac{1}{2}}}$, $\dfrac{ay\dot{y}}{\left(y^2 + \dfrac{3a^2}{4}\right)^{\frac{1}{2}}}$, and

$\dfrac{\dfrac{a^2}{4}\dot{y}}{\left(y^2 + \dfrac{3a^2}{4}\right)^{\frac{1}{2}}}$, of which the first is known by Fluent 14.;

the second $= a.\left(y^2 + \dfrac{3a^2}{4}\right)^{\frac{1}{2}}$, and the third $= \dfrac{a^2}{4} \times$ hyp. log.

of $\left(y + \left(y^2 + \dfrac{3a^2}{4}\right)^{\frac{1}{2}}\right)$.

RULE VIII. If the fluxional quantity without the tri-
nomial be not properly related to the fluxion of the
highest power of the variable quantity in it, after sub-
stitution the binomial may be expanded, and the fluent
of each term taken separately. If the series terminates,
the whole fluent is obtained; if not, this is only an
approximation to its value.

FLUENT 84.

· (285.) Required the fluent of $\dfrac{z^{m}\dot{z}}{a+bz+cz^{2}}$.

Let $z + \dfrac{b}{2c} = x$; then $\dot{z}=\dot{x}$, and $z^{m} = \left(x-\dfrac{b}{2c}\right)^{m}$.

Also $z^{2} + \dfrac{bz}{c} + \dfrac{a}{c} = x^{2} \pm d^{2}$, if $\pm d^{2} = \dfrac{a}{c} - \dfrac{b^{2}}{4c}$, and the

fluxion $= \dfrac{\dot{x} \cdot \left(x-\dfrac{b}{2c}\right)^{m}}{c \cdot (x^{2} \pm d^{2})}$; where m being by supposition

an affirmative integer, $\left(x-\dfrac{b}{2c}\right)^{m}$ must be expanded,
and the fluent of each term be taken by continuation.

This is of the same form with the 4th of De Moivre's,
Coroll. 1.

FLUENT 85.

(286.) To find the fluent of $\dfrac{z^{-m}\dot{z}}{a+bz+cz^{2}}$.

Let $\dfrac{1}{y}=z$; $\therefore \dot{z}=\dfrac{-\dot{y}}{y^{2}}$, and $z^{-m}=y^{m}$, and $a+bz+cz^{2}$

$=a+\dfrac{b}{y}+\dfrac{c}{y^{2}}$; therefore the fluxion $=\dfrac{-y^{m}\dot{y}}{ay^{2}+by+c}$, whose
fluent may be found by the last case.

This is of the same form with De Moivre's 7th,
Coroll. 1.

FLUENT 86.

(287.) Required the fluent of $\dfrac{z^{rn-1}\dot{z}}{a+bz^n+cz^{2n}}$.

Let $z^n=y$; then $z^{rn}=y^r$, and $z^{rn-1}\dot{z}=\dfrac{y^{r-1}\dot{y}}{n}$; the denominator becomes $a+by+cy^2$, and the whole fluxion $=$

$\dfrac{\frac{1}{n}\times y^{r-1}\dot{y}}{a+by+cy^2}$; in which $r-1$ being assumed an affirmative integer, the fluent can be found by Fluent 84.

FLUENT 87.

(288.) To find the fluent of $\dfrac{z^{-rn-1}\dot{z}}{a+bz^n+cz^{2n}}$.

Let $z^n=\dfrac{1}{y}$; then $z^{-rn}=y^r$; and $z^{-rn-1}\dot{z}=-\dfrac{y^{r-1}\dot{y}}{n}$.

Also the denominator $=a+\dfrac{b}{y}+\dfrac{c}{y^2}$; and the whole fluxion $=$ $-\dfrac{\frac{1}{n}\times y^{r+1}\dot{y}}{ay^2+by+c}$, whose fluent can be found by Fluent 84.

This and the last are of a form similar to the 12th of De Moivre's.

FLUENT 88.

(289.) To find the fluent of $\dfrac{z^m\dot{z}}{(a+bz+cz^2)^{\frac{1}{2}}}$, m being an integer.

Assume $z+\dfrac{b}{2c}=x$; then $\dot{z}=\dot{x}$, and $z^m=\left(x-\dfrac{b}{2c}\right)^m$;

and the whole may be reduced to the form
$\dfrac{\left(x-\dfrac{b}{2c}\right)^{m} \times \dot{x}}{c^{\frac{1}{2}}.(x^{2}\pm d^{2})^{\frac{1}{2}}}$; whose fluent may be found by expanding the binomial.

FLUENT 89.

(290.) 1. To find the fluent of $\dfrac{z^{-m}\dot{z}}{(a+bz+cz^{2})^{\frac{1}{2}}}$.

Let $z=\dfrac{1}{y}$, then $z^{-m}=y^{m}$, and $\dot{z}=\dfrac{-\dot{y}}{y^{2}}$; hence the

fluxion becomes, by the same methods, $=\dfrac{-y^{m-1}\dot{y}}{(ay^{2}+by+c)^{\frac{1}{2}}}$,

whose fluent may be found by the last.

2. The fluent of $\dfrac{z^{rn-1}\dot{z}}{(a+bz^{n}+cz^{2n})^{\frac{1}{2}}}$ may be found in

the same manner, by assuming $y=z^{n}$.

3. And that of $\dfrac{z^{-rn-1}\dot{z}}{(a+bz^{n}+cz^{2m})^{\frac{1}{2}}}$, by taking $z^{n}=\dfrac{1}{y}$.

FLUENT 90.

(291.) 1. For the fluent of $z^{m}\dot{z}.(a+bz+cz^{2})^{\frac{1}{2}}$, take $x=$

$z+\dfrac{b}{2c}$; it may be thus reduced to the form $\left(x-\dfrac{b}{2c}\right)^{m}\times$

$\dot{x}c^{\frac{1}{2}}(x^{2}+d^{2})^{\frac{1}{2}}$, the fluent of which is found by expanding

$\left(x-\dfrac{b}{2c}\right)^{m}$.

2. For the fluent of $z^{-m}\dot{z}.(a+bz+cz^{2})^{\frac{1}{2}}$, assume $z=$

$\dfrac{1}{y}$ the given fluxion $=-\dot{y}y^{m-3}.(ay^{2}+by+c)^{\frac{1}{2}}$, whose

fluent is found by the last case.

3. For that of $x^{m-1}\dot{x}.(a+bx^n+cx^{2n})^{\frac{1}{2}}$, take x^n also $=y$.

4. And for that of $x^{-m-1}\dot{x}\,(a+bx^n+cx^{2n})^{\frac{1}{2}}$, let $x^n=\dfrac{1}{y}$.

FLUENT 91.

(292.) To find the fluent of $\dfrac{x\dot{x}}{(2x-4nx+2nx^2)^{\frac{1}{2}}}$.

This $=x\dot{x}.(2x-4nx+2nx^2)^{-\frac{1}{2}}$. Assume $2x=a$, $4nx-2nx^2=b$; then since $(a-b)^m=a^m-ma^{m-1}b+m.\dfrac{m-1}{2}a^{m-2}b^2-$&c. by substituting for a, b and m their values,

$$(2x-4nx+2nx^2)^{-\frac{1}{2}}=\frac{x^{-\frac{1}{2}}}{2^{\frac{1}{2}}}+\frac{nx^{-\frac{1}{4}}}{2^{\frac{1}{2}}}-\frac{nx^{\frac{1}{4}}}{2^{\frac{3}{2}}}+\text{ \&c.; and}$$

the fluxion $=\dfrac{x^{\frac{1}{2}}\dot{x}}{2^{\frac{1}{2}}}+\dfrac{nx^{\frac{1}{4}}\dot{x}}{2^{\frac{1}{2}}}-\dfrac{nx^{\frac{3}{4}}\dot{x}}{2^{\frac{3}{2}}}+\text{\&c.; whose fluent}=$

$$\frac{2^{\frac{1}{2}}x^{\frac{3}{2}}}{3}+\frac{2^{\frac{1}{2}}nx^{\frac{3}{2}}}{3}-\frac{nx^{\frac{5}{4}}}{5\times 2^{\frac{1}{2}}}+\text{\&c.}$$

This fluent is used in Art. 138. Prob. 9. in the Attraction of Bodies.

FLUENT 92.

(293.) To find the fluent of $\dfrac{x\dot{x}}{(2x+4nx-2nx^2)^{\frac{1}{4}}}$.

Assume $a=2x$; $b=4nx-2nx^2$; then since $(a+b)^m=a^m+ma^{m-1}b+m.\dfrac{m-1}{2}a^{m-2}b^2+$&c. by substituting

as in the last case, we have $x\,\dot{x}.\,(2x+4nx-2nx^2)^{-\frac{1}{2}}=$

$$\dfrac{x^{\frac{1}{2}}\dot{x}}{2^{\frac{1}{2}}}+\dfrac{nx^{\frac{3}{2}}\dot{x}}{2^{\frac{3}{2}}}-\dfrac{nx^{\frac{1}{2}}\dot{x}}{2^{\frac{1}{2}}}+\&c.\,;\ \text{and the fluent}=\dfrac{2^{\frac{1}{2}}x^{\frac{3}{2}}}{3}+$$

$$\dfrac{n\,x^{\frac{5}{2}}}{5\times 2^{\frac{1}{2}}}-\dfrac{2^{\frac{1}{2}}nx^{\frac{3}{2}}}{3}+\&c.$$

This fluent is used in Art. 138. Prob. 10. on the Attraction of Bodies. In both these cases, if n be very small, the succeeding terms vanish, and the whole fluent is expressed by the three first terms of the series.

FLUENT 93.

(294.) To find the fluent of $\dfrac{a\dot{y}}{(a^2-y^2)^{\frac{1}{2}}}$ in a series.

In cases of this kind, no substitution is necessary.

For $a^2-y^2=a^2.\left(1-\dfrac{y^2}{a^2}\right);\ \therefore\ \dfrac{a\dot{y}}{(a^2-y^2)^{\frac{1}{2}}}=\dfrac{\dot{y}}{\left(1-\dfrac{y^2}{a^2}\right)^{\frac{1}{2}}}=$

$\dot{y}\times\left(1-\dfrac{y^2}{a^2}\right)^{-\frac{1}{2}};$ which, expanded by the series $(a-b)^m$,

$=\dot{y}.\left(1+\dfrac{y^2}{2a^2}+\dfrac{3y^4}{2.4a^4}+\&c.\right)=\dot{y}+\dfrac{y^2\dot{y}}{2a^2}+\dfrac{3y^4\dot{y}}{2.4a^4}+\&c.\,;$

and the fluent $=y+\dfrac{y^3}{2.3.a^2}+\dfrac{3y^5}{2.4.5.a^4}+\&c.$

This fluent is used in Art. 55. Ex. 5.

(295.) RULE IX. When the denominator is a rational trinomial or multinomial, the best way of proceeding in general is to resolve the given fraction into binomial ones, by assuming the denominator$=0$, and finding its roots.

FLUENT 94.

(296.) To find the fluent of $\dfrac{\dot{x}}{x^2+ax+b}$.

Let $x^2+ax+b=0$; then the roots are $-\dfrac{a}{2}-\left(\dfrac{a^2}{4}-b\right)^{\frac{1}{2}}$,

and $-\dfrac{a}{2}+\left(\dfrac{a^2}{4}-b\right)^{\frac{1}{2}}$. Let these quantities $= p$ and q; then $(x-p).(x-q) = x^2 + ax + b$.

Let then $\dfrac{A}{x-p} + \dfrac{B}{x-q} = \dfrac{1}{x^2 + ax + b}$; then

$\dfrac{A.(x-q)+B.(x-p)}{(x-p).(x-q)} = \dfrac{1}{x^2+ax+b}$; hence, the nume-

rators being equal as well as the denominators, we have

$$\left.\begin{array}{c} Ax - Aq \\ + Bx - Bp \\ - 1 \end{array}\right\} = 0\ ;$$

$\therefore Ax+Bx=0$, and $A=-B$; also $-Aq-Bp=1$; or,

since $A = -B$, $Ap-Aq = 1$; $\therefore A=\dfrac{1}{p-q}$, and $B=$

$\dfrac{1}{q-p}$; $\therefore$ since $\dfrac{A\dot{x}}{x-p} + \dfrac{B\dot{x}}{x-q} = \dfrac{\dot{x}}{x^2+ax+b}$, we have

$\dfrac{\dot{x}}{(p-q).(x-p)} + \dfrac{\dot{x}}{(q-p).(x-q)} = \dfrac{\dot{x}}{x^2+ax+b}$; and

the fluent $= \dfrac{1}{p-q} \times$ hyp. log. $(x-p) + \dfrac{1}{q-p} \times$ hyp. log. $(x-q)$.

(297.) The quantities A, B, &c. may be determined more easily, by substituting one of its roots for x after the multiplication, in which case most of the terms will vanish.

Thus, let p, q, r, &c. be roots of the equation

$x^n - Px^{n-1} + \&c. = 0$, to resolve the fraction $\dfrac{1}{x^n - Px^{n-1} + \&c.}$

into $\dfrac{K}{x-p} + \dfrac{L}{x-q} + \dfrac{M}{x-r} + \&c.$

Reduce the fractions to a common denominator; then the denominators on both sides of this equation will be equal, and the sum of the numerators will $= 1$. That is, $K.(x - q).(x - r).\&c. + L.(x - p).(x - r).\&c. + M.(x-p).(x-q).\&c. = 1$. Now as this is true for every value of x, let $x = p$; then $x - p = 0$; $\therefore K$ $(p-q).(p-r).\&c. = 1$, and $K = \dfrac{1}{(p-q).(p-r).\&c.}$. Next let $x = q$; then $x - q = 0$, and $L.(q-p).(q-r).\&c.$ $= 1$; $\therefore L = \dfrac{1}{(q-p).(q-r).\&c.}$; and in the same manner the other numerators may be determined.

FLUENT 95.

(298.) To find the fluent of $\dfrac{x^2 \dot{x}}{x^3 + ax^2 + bx + c}$.

Assume p, q, and r, the roots; and let $\dfrac{K}{x-p} + \dfrac{L}{x-q} +$

$\dfrac{M}{x-r} = \dfrac{1}{x^3 + ax^2 + bx + c}$; then $K = \dfrac{1}{(p-q).(p-r)}$;

$L = \dfrac{1}{(q-p).(q-r)}$; $M = \dfrac{1}{(r-p).(r-q)}$; and the

given fluxion $= \dfrac{Kx^2 \dot{x}}{x-p} + \dfrac{Lx^2 \dot{x}}{x-q} + \dfrac{Mx^2 \dot{x}}{x-r}$; of which the

fluents, by actual division, are $K \times \dfrac{x^2}{2} + Kpx + Kp^2 \times$

hyp. log. $(x - p) + L \times \dfrac{x^2}{2} + Lqx + Lq^2 \times$ hyp. log

$(x-q) + M \times \dfrac{x^2}{2} + Mrx + Mr^2 \times$ hyp. log. $(x-r) =$

$(K+L+M).\dfrac{x^2}{2} + (Kp+Lq+Mr).x+Kp^2 \times$ hyp. log. $(x-p) + Lq^2 \times$ hyp. log. $(x-q) + Mr^2 \times$ hyp. log. $(x-r)$.

FLUENT 96.

(299.) To find the fluent of $\dfrac{x^m \dot{x}}{x^n - Px^{n-1}+\&c.}$, m being a whole positive number, and the roots of the denominator p, q, r, &c.

This fluxion may be resolved like the last, into the following quantities; $\dfrac{Kx^m \dot{x}}{x-p} + \dfrac{Lx^m \dot{x}}{x-q} + \&c.$; where K, L, &c. may be determined as before, and the fluents may be found in each case by actual division. The first fluent

$= \dfrac{Kx^m}{m} + \dfrac{Kpx^{m-1}}{m-1} + \&c. + Kp^m \times$ hyp. log. $(x-p)$;

in like manner the fluents of the other quantities are found. The sum of all these is the fluent required; and it $= (K + L + \&c.) \times \dfrac{x^m}{m} + (Kp + Lq + \&c.)$

$\dfrac{x^{m-1}}{m-1} + \&c. + Kp^m \times$ hyp. log. $(x-p) + Lq^m \times$ hyp. log. $(x-q) + \&c.$

To determine when any of these coefficients become equal to nothing, Vide Dr. Waring's *Medit. Algeb.* in the Addenda.

(300.) If two roots, as p and q, are equal, one of the quantities must have a quadratic divisor $(x-p)^2$.

Let $\dfrac{1}{x^3 - ax^2 + bx - c} = \dfrac{Lx + M}{(x-p)^2} + \dfrac{N}{x-r}$; reduce them to a common denominator, and equate the numerators;

then $Lx^2 - Lrx - Mr$
$+ Nx^2 + Mx + Np^2 \Big\} = 0;$
$- 2Npx - 1$

hence, $L+N=0; M-Lr-2Np=0; -Mr+Np^2-1=0;$

$\therefore L=-N; M=\dfrac{Np^2-1}{r}; \therefore \dfrac{Np^2-1}{r}+Nr-2Np=0;$

$\therefore N=\dfrac{1}{(p-r)^2}; L=\dfrac{1}{-(r-p)^2}; M=\dfrac{2p-r}{(p-r)^2}.$

Hence the fluent of $\dfrac{\dot{x}}{x^3-ax^2+bx-c}=f.\dfrac{Lx\dot{x}+M\dot{x}}{(x-p)^2}+$

$f.\dfrac{N\dot{x}}{x-r}$; where $L, M,$ and N are found. And this fluent

$= L \times$ hyp. log. $(x-p) - \dfrac{Lp+M}{x-p} + N \times$ hyp. log.
$(x-r).$ (Fluent 54.)

(301.) If some of the roots $p, q, r,$ &c. be impossible, the fractions, in which the impossible roots are contained, must be incorporated in pairs; and then the impossible quantities will disappear. For if $\pm a + \sqrt{-b^2}$, and $\pm a - \sqrt{-b^2}$ be the roots of an equation of two dimensions, since they enter by pairs, the equation itself is $\left(x \mp a - \sqrt{-b^2}\right).\left(x \mp a + \sqrt{-b^2}\right)$, or $x^2 \mp 2ax + a^2 + b^2 = 0$; where the imaginary quantities are not found.

FLUENT 97.

(302.) Let the proposed fraction be $\dfrac{\dot{x}}{x^3+ax^2+bx+c}$, and let two roots, p and $q,$ of the cubic equation be impossible.

Assume $\dfrac{L}{x-p} + \dfrac{M}{x-q} + \dfrac{N}{x-r} = \dfrac{1}{x^3+ax^2+bx+c}$,

and incorporate the two first; then we have the given

fluxion $= \dfrac{(L + M) \cdot x\dot{x} - (Lq + Mp) \cdot \dot{x}}{x^2 - (p + q) \cdot x + pq} + \dfrac{N\dot{x}}{x - r}$;

where the impossible parts vanish, as will appear by substituting $a + \sqrt{-b^2}$ and $a - \sqrt{-b^2}$ for p and q. The fluent of the first part is found by the method of Fluent 94.; and the fluent of the last is $N \times$ hyp. log. $(x - r)$.

FLUENT 98.

(303.) In general, to find the fluent of $\dfrac{x^m\dot{x}}{x^2 - ax + b}$.

If the roots be both possible, resolve $\dfrac{1}{x^2 - ax + b}$ into

$\dfrac{K}{x - p} + \dfrac{L}{x - q}$; and the given fluxion $= \dfrac{Kx^m\dot{x}}{x - p} + \dfrac{Lx^m\dot{x}}{x - q}$, whose fluents have already been found.

But if the roots be impossible, divide $x^m\dot{x}$ by $x^2 - ax + b$, until the remainder becomes $cx\dot{x} - d\dot{x}$, c and d being the coefficients, which arise from the division; let the quotient be $x^{m-2}\dot{x} + ax^{m-3}\dot{x} + (a^2 - b) \cdot$ $x^{m-4}\dot{x} + \&c. + \dfrac{cx\dot{x} - d\dot{x}}{x^2 - ax + b}$; and the fluent $= \dfrac{x^{m-1}}{m-1} +$

$\dfrac{ax^{m-2}}{m-2} + \dfrac{(a^2 - b) \cdot x^{m-3}}{m-3} + \&c. + f. \dfrac{cx\dot{x} - d\dot{x}}{x^2 - ax + b}$, or $+ \frac{1}{2}c \times$ hyp. log. $(x^2 \pm e^2) + A$. (Fluent 81.)

(304.) RULE X. When the denominator of the fluxion is of the form $(x + a)^m \cdot (x + b)^n$, the given fraction must be resolved into others of the form $\dfrac{L}{(x+a)^m} +$

$\dfrac{M}{(x+a)^{m-1}} + \dfrac{N}{(x+a)^{m-2}} + \&c. + \dfrac{P}{(x+b)^n} + \dfrac{Q}{(x+b)^{n-1}} +$

$\dfrac{R}{(x+b)^{n-2}}$ + &c., respectively continued to m and n quantities.

(305.) LEMMA. To resolve $\dfrac{1}{(x+a)^m \cdot (x+b)^n}$ into

$\dfrac{L}{(x+a)^m} + \dfrac{M}{(x+a)^{m-1}}$ + &c.

Reduce the fractions to a common denominator, and suppose the numerators on each side equal; then $L \cdot (x+b)^n + M \cdot (x+b)^n \cdot (x+a) + N \cdot (x+b)^n \cdot (x+a)^2$ + &c. + $P \cdot (x+a)^m + Q \cdot (x+a)^m \cdot (x+b) + R \cdot (x+a)^m \cdot (x+b)^2$ + &c. = 1. Call this equation A. Now assume $x + a = 0$; then $x = -a$, and all those terms, into which $x + a$ enters, $= 0$; $\therefore$ $L \cdot (b-a)^n = 1$, and $L = \dfrac{1}{(b-a)^n}$. Take the fluxion of the equation A, and divide by $\dot{x}$; then $nL \cdot (x+b)^{n-1} + nM \cdot (x+b)^{n-1} \times (x+a) + M \cdot (x+b)^n$ + &c. $= 0$. Let $x = -a$, and we have $nL \cdot (b-a)^{n-1} + M \cdot (b-a)^n = 0$; $\therefore$ $M = \dfrac{-nL}{b-a} = \dfrac{-n}{(b-a)^{n+1}}$. Continue to take the fluxion of the last equation, making $x + a = 0$, and the values of N, &c. will be determined.

Again, assume $x + b = 0$, and P will be found $= \dfrac{1}{(a-b)^m}$; take the fluxion, and again assume $x + b = 0$; then $Q = \dfrac{-m}{(a-b)^{m+1}}$, and so on for R, S, &c.

FLUENT. 99.

(306.) To find the fluent of $\dfrac{x^r \dot{x}}{(x+a)^m \cdot (x+b)^n}$.

By the last Article, the given fluxion $= \dfrac{Lx^r\dot{x}}{(x+a)^m} +$

$\dfrac{Mx^r\dot{x}}{(x+a)^{m-1}} + \&c. + \dfrac{Px^r\dot{x}}{(x+b)^n} + \dfrac{Qx^r\dot{x}}{(x+b)^{n-1}} + \&c.$ Assume

$x+a=z$; then $x^{r+1} = (z-a)^{r+1}$, and $x^r\dot{x} = \dot{z}.(z-a)^r$;

$\therefore \dfrac{Lx^r\dot{x}}{(x+a)^m} = L \times \dfrac{(z-a)^r\dot{z}}{z^m} =$ by actual expansion,

$L \times (z^{r-m}\dot{z} - raz^{r-m-1}\dot{z} + r.\dfrac{r-1}{2} \times a^2z^{r-m-2}\dot{z} - \&c.$

to $r+1$ terms). The fluent of every term in this series can be found by the common method, except that where the index of z is -1, which involves the logarithm of z. In the same manner the fluents of the other series can be determined, and their sum is the fluent required.

SECTION V.

(307.) The sixteen following Fluents are taken from De Moivre's *Miscellanea Analytica*, Lib. III.

### FLUENT 100. (*De Moivre* 1.)

To find the fluent of $\dfrac{\dot{z}}{1+2az+z^2}$ (A), where a is less than 1, or the roots of the denominator impossible.

Let $v = z + a$; then $1 + 2az + z^2 = v^2 - a^2 + 1 = s^2 + v^2$, if $1 - a^2 = s^2$; and $\dot{z} = \dot{v}$; $\therefore$ the given fluxion =

$$\frac{\dot{v}}{s^2+v^2} = \frac{1}{s^2} \times \frac{s^2\dot{v}}{s^2+v^2};$$ and the fluent $= \frac{1}{s^2} \times$ a circular

arc of radius s, and tangent v; or of radius $= (1-a^2)^{\frac{1}{2}}$, and tangent $z+a$.

Hence this construction :

With C as center, and $CE = 1$ as radius, describe the semicircle EDF. Take $CA = z$, and CB on the other side of $C = a$; draw the ordinate BD; with D as center, and DB radius, describe a circular arc BGH; then $BA = z + a = v$; and $DB = s = (1-a^2)^{\frac{1}{2}}$; $\therefore$ the fluent $= \dfrac{1}{DB^2} \times$ the arc BH.
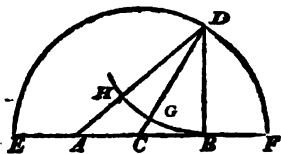
(308.) The fluent of $\dfrac{\dot{z}}{1-2az+z^2}$ is found by taking

$z - a = v;$ and it $= \dfrac{1}{s^2} \times$ a circular arc of radius s, and

tangent $z - a$.

Here take $CA = z$, $CB = a$ on the same side of C, and draw the ordinate BD; join DC, DA; then the fluent $= \dfrac{1}{DB^2} \times$ the arc BH.
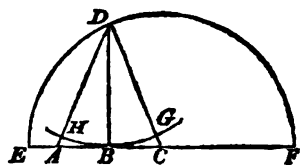
(309.) The fluent of $\dfrac{\dot{z}}{r^2 + 2arz + z^2}$, where a is less

than 1, may be found in the same manner by assuming

$\dfrac{z}{r} + a = v;$ for in this case $r^2 + 2arz + z^2 = r^2 . (s^2 + v^2);$

and $\dot{z} = r\dot{v};$ $\therefore$ the given fluxion $= \dfrac{r\dot{v}}{r^2 . (s^2 + v^2)} = \dfrac{1}{rs^2} \times$

$\dfrac{s^2 \dot{v}}{s^2 + v^2}$; and the fluent $= \dfrac{1}{rs^2} \times$ a circular arc of radius $s =$

$(1 - a^2)^{\frac{1}{2}}$, and tangent $\dfrac{z}{r} + a$.

(310.) And the fluent of $\dfrac{\dot{z}}{r^2 - 2arz + z^2}$, where a is

less than 1, may be found by taking $\dfrac{z}{r} - a = v$.

FLUENT 101.   (*De Moivre* 2.)

(311.) To find the fluent of $\dfrac{z \dot{z}}{1 + 2az + z^2}$ ($\dot{B}$), where

a is less than 1.

Let $v = z + a$; then the denominator $= s^2 + v^2$; also
$z^2 = v^2 - 2av + a^2$; $\therefore z \dot{z} = v \dot{v} - a \dot{v}$; and the given

fluxion $= \dfrac{v \dot{v} - a \dot{v}}{s^2 + v^2}$; and the fluent $= \frac{1}{2}$ hyp. log. $(s^2 + v^2)$

$- \dfrac{a}{s} \times$ a circular arc of radius s, and tangent $v =$ hyp. log.

$(s^2 + v^2)^{\frac{1}{2}} - \dfrac{a}{s} \times$ arc BH (Fig. to Fluent 100.) $=$ hyp.

log. of $AD - \dfrac{CB}{DB^2} \times$ arc BH.

(312.) The fluent of $\dfrac{z \dot{z}}{1 - 2az + z^2}$ is found by as-

suming $z - a = v$.

(313.) The fluent of $\dfrac{z \dot{z}}{r^2 + 2arz + z^2}$, where a is less

than 1, may be found by assuming $\dfrac{z}{r} + a = v$;

And of $\dfrac{z \dot{z}}{r^2 - 2arz + z^2}$, by taking $\dfrac{z}{r} - a = v$.

FLUENT 102.   (*De Moivre* 3.)

(314.) To find the fluent of $\dfrac{x^2 \dot{x}}{1 + 2ax + x^2}$, ($\dot{C}$), where a is less than 1.

Divide the numerator by the denominator in an inverted order.

$$x^2 + 2ax + 1) \, x^2 \dot{x} \, (\dot{x}$$
$$\underline{\qquad x^2 \dot{x} + 2ax \dot{x} + \dot{x}}$$
$$- \, 2ax\dot{x} - \dot{x}$$

$\therefore \dot{C} = \dot{x} - \dfrac{2ax\dot{x}}{1 + 2ax + x^2} - \dfrac{\dot{x}}{1 + 2ax + x^2}$; and the fluent or $C = x - 2aB - A$.

FLUENT 103.   (*De Moivre* 4.)

(315.) To find the fluent of $\dfrac{x^3 \dot{x}}{1 + 2ax + x^2}$ ($\dot{D}$), a being less than 1.

Divide, as before, in an inverted order ; then the given fluxion $= x\dot{x} - \dfrac{2ax^2 \dot{x}}{1 + 2ax + x^2} - \dfrac{x\dot{x}}{1 + 2ax + x^2}$;

or $\dot{D} = x\dot{x} - 2a\dot{C} - \dot{B}$;

$\therefore D = \dfrac{x^2}{2} - 2aC - B$.

In the same manner the series might be continued to any power of x.

(316.) COR. 1. Hence, if a be less than 1, and m any whole positive number, we can determine the fluent of $\dfrac{x^m \dot{x}}{1 + 2ax + x^2}$.

Divide the numerator by the denominator in an inverted order, till the remainder becomes $\pm \dfrac{c z \dot{z} \pm e \dot{z}}{1 + 2 a z + z^{2}}$, c and e being the coefficients; then the given fluxion $=$

$$z^{m-2} \dot{z} - 2 a z^{m-3} \dot{z} + 2 b z^{m-4} \dot{z} - \&c. \pm \frac{c z \dot{z} \pm e \dot{z}}{1 + 2 a z + z^{2}}; \text{ and}$$

the required fluent $= \dfrac{z^{m-3}}{m-3} - \dfrac{2 a z^{m-4}}{m-4} + \&c. \pm c B \pm e A.$

(317.) Cor. 2. If a be less than 1, and m be a positive fraction, the fluent of $\dfrac{z^{m} \dot{z}}{1 + 2 a z + z^{2}}$ can be found.

Let $m = \dfrac{d}{t}$, a fraction in its lowest terms; then, if

$v = z^{\frac{1}{t}}$, $v^{d} = z^{\frac{d}{t}}$; also $z = v^{t}$; $\therefore \dot{z} = t v^{t-1} \dot{v}$, and $z^{2} = v^{2t}$; hence the given fluxion $= \dfrac{t v^{d + t - 1} \dot{v}}{1 + 2 a v^{t} + v^{2t}}$. But (by Prob.

77.) the fraction $\dfrac{1}{1 + 2 a v^{t} + v^{2t}}$ is divisible into component

parts of this form $\dfrac{p - q v}{1 - 2 a v + v^{2}}$; therefore this fluxion is

reducible to the form $\dfrac{p t v^{d + t - 1} \dot{v}}{1 - 2 b v + v^{2}} - \dfrac{q t v^{d + t} \dot{v}}{1 - 2 b v + v^{2}}$, of

which the fluents are known by the last Corollary; for here b is less than 1, and d and t are whole numbers.

Fluent 104.   (*De Moivre* 5.)

(318.) To find the fluent of $\dfrac{z^{-1} \dot{z}}{1 + 2 a z + z^{2}} (\dot{E})$, a being less than unity.

Divide the numerator by the denominator in the

natural order; then $\dot{E} = \dfrac{\dot{z}}{z} - \dfrac{2a\dot{z}}{1 + 2az + z^2} - \dfrac{z\dot{z}}{1 + 2az + z^2} =$

$\dfrac{\dot{z}}{z} - 2a\dot{A} - \dot{B}$; and the fluent = hyp. log. of $z - 2aA - B$.

FLUENT 105.   (*De Moivre* 6.)

(319.) To find the fluent of $\dfrac{z^{-2}\dot{z}}{1 + 2az + z^2}$ $(\dot{F})$, a being less than unity.

Divide as before; then $\dot{F} = \dfrac{\dot{z}}{z^2} - 2a\dot{E} - \dot{A}$; and the

fluent $= -\dfrac{1}{z} - 2aE - A$.

FLUENT 106.   (*De Moivre* 7.)

(320.) To find the fluent of $\dfrac{z^{-3}\dot{z}}{1 + 2az + z^2}$ $(\dot{G})$, a being less than unity.

By division, this fluxion $= z^{-3}\dot{z} - 2a\dot{F} - \dot{E}$; $\therefore$ the

fluent $G = -\dfrac{1}{2z^2} - 2aF - E$; and the fluents may be continued to any power.

(321.) COR. 1. If m be any whole number, the fluent of $\dfrac{z^{-m}\dot{z}}{1 + 2az + z^2}$ can be found by the same method.

Or if $v = \dfrac{1}{z}$, it can be transformed into the fluxion

$\dfrac{-v^m\dot{v}}{1 + 2av + v^2}$, of which the fluent is known by Art. 316.

(322.) COR. 2. And if m be a fraction, and a less

than 1, the fluent can be found by the method adopted in Art. 317.

FLUENT 107.   (*De Moivre* 8.)

(323.) The fluent of $\dfrac{\dot{z}}{1 + 2az + z^2}$, where a is greater than 1, is of the same form with the Fluent 94, and found in the same manner.

FLUENT 108.   (*De Moivre* 9.)

(324.) To find the fluent of $\dfrac{z^n \dot{z}}{1 + mz}$, n being a whole positive number.

Divide the numerator by the denominator in an inverted order; then the fluent $= \dfrac{z^n}{mn} - \dfrac{z^{n-1}}{m^2 \cdot (n-1)} +$

$\dfrac{z^{n-2}}{m^3 \cdot (n-2)} -$ &c. $(n) \pm \dfrac{1}{m^{n+1}} \times$ hyp. log. $(1 + mz)$.

This is similar to Fluent 4.

FLUENT 109.   (*De Moivre* 10.)

(325.) The fluent of $\dfrac{z^{-n} \dot{z}}{1 + mz}$ is found by dividing

in the natural order; and it $= -\dfrac{1}{(n-1) \cdot z^{n-1}} + \dfrac{m}{(n-2) \cdot z^{n-2}} -$

&c. $(n) \pm m^{n-1} \times$ hyp. log. $(1 + mz)$.

FLUENT 110.   (*De Moivre* 11.)

(326.) To find the fluent of $\dfrac{z^{\frac{d}{i}} \dot{z}}{1 + mz}$.

Take $y = (mz)^{\frac{1}{i}}$; $\therefore mz = y^i$; also $z^{\frac{1}{i}} = \dfrac{y}{m^{\frac{1}{i}}}$, and $z^{\frac{d+i}{i}} =$

$\frac{y^{d+t}}{m^{\frac{d+t}{t}}}$, $\therefore \frac{d+t}{t} \times z^{\frac{d}{t}} \dot{z} = \frac{(d+t) . y^{d+t-1} \dot{y}}{m^{\frac{d+t}{t}}}$, and $z^{\frac{d}{t}} \dot{z} = t \times$

$\frac{y^{d+t-1} \dot{y}}{m^{\frac{d+t}{t}}}$; but $1 + mz = 1 + y^t$; $\therefore$ the given fluxion is

equal to $\frac{t y^n \dot{y}}{m^{\frac{n+1}{t}} \times (1+y^t)}$, n being substituted for $d+t-1$.

Now by Prob. 80. and 81. $\frac{1}{1+y^t}$ can be resolved into

quadratic divisors of the form $\frac{q - ry}{1 - 2ay + y^2}$, &c. where

t is even, and a is less than 1; therefore the resulting

fluxions in this case will be of the form $\frac{t}{m^{\frac{n+1}{t}}} \times \frac{q y^n \dot{y} - r y^{n+1} \dot{y}}{1 - 2ay + y^2}$,

whose fluents can be found by the preceding Cases.

If t is odd, the first divisor is of the form $\frac{p}{1 + y}$,

and the first resulting fluxion is $\frac{t}{m^{\frac{n+1}{t}}} \times \frac{p y^n \dot{y}}{1 + y}$, whose

fluent can be found.

(327.) In the same manner the fluent of $\frac{z^{\frac{d}{t}} \dot{z}}{1 - mz}$ can

be found by the resolutions in Prob. 78. and 79.

FLUENT 111.   (*De Moivre* 12.)

(328.) To find the fluent of $\frac{z^m \dot{z}}{1 + 2lz^n + z^{2n}}$, where l

is less than unity.

By Prob. 77. $\frac{1}{1 + 2lz^n + z^{2n}}$ can be resolved into quad-

ratic divisors of the form $\dfrac{p-qz}{1-2az+z^2}$, where a is less

than 1; $\therefore$ by multiplying $z^m \dot{z}$ into these divisors, the

resulting fluxions will be of the form $\dfrac{pz^m \dot{z} - qz^{m+1}\dot{z}}{1-2az+z^2}$,

of which the fluents are known by the preceding Cases,
whatever be the value of m.

(329.) If the given fluxion be $\dfrac{z^m \dot{z}}{1 + 2lz^{-n} + z^{-2n}}$,

multiply both the numerator and the denominator by z^{2n},

and the fluxion will become $\dfrac{z^{m+2n}\dot{z}}{z^{2n}+2lz^n + 1}$, whose fluent

can be found.

FLUENT 112. (*De Moivre* 13.)

(330.) To find the fluent of $\dfrac{x^{\frac{d}{i}} \dot{x}}{1 \pm 2lx^{\frac{n}{p}} + x^{\frac{2n}{p}}}$.

Take $x^{\frac{1}{p}} = y$; $\therefore x^{\frac{1}{i}} = y^p$, and $x^{\frac{d+i}{i}} = y^{(d+i)\cdot p}$; hence

$\dfrac{d+t}{t} \times x^{\frac{d}{i}}\dot{x} = (d+t)\, p y^{dp+p-1}\dot{y}$, and $x^{\frac{d}{i}}\dot{x} = tp \times y^{dp+p-1}\dot{y}$.

Again, $x^{\frac{1}{p}} = y$; $\therefore x^{\frac{n}{p}} = y^n$; hence the given fluxion

becomes $= \dfrac{tp y^{dp+p-1}\dot{y}}{1 \pm 2ly^n + y^{2n}}$, whose fluent is known by the

preceding Cases.

FLUENT 113. (*De Moivre* 14.)

(331.) To find the fluent of $\dfrac{z^m \dot{z}}{1 \pm z^n}$.

By Prob. 78, 79, 80, and 81, the quantity $\dfrac{1}{1 \pm z^n}$

can be resolved into divisors, either all trinomials of the

form $\dfrac{q - rz}{1 - 2az + z^2}$, where a is less than unity; or into divisors partly trinomial and partly binomial, of the form $\dfrac{p}{1 - z^2}$, or $\dfrac{p}{1 - z}$; $z^m \dot{z}$ is to be multiplied into these divisors, and the fluents can be found as before.

(332.) The fluent of $\dfrac{z^m \dot{z}}{r^n \pm z^n}$ can be found in the same manner by the resolution of $\dfrac{1}{r^n \pm z^n}$.

Fluent 114.　　(*De Moivre* 15.)

(333.) To find the fluent of $\dfrac{z^m \dot{z}}{1 + 2lz^n + z^{2n}}$, where l is greater than 1.

Let $z^n = x$; then $z^{2n} = x^2$; also $z = x^{\frac{1}{n}}$; $\therefore z^{m+1} = x^{\frac{m+1}{n}}$, and $z^m \dot{z} = \dfrac{x^{\frac{m-n+1}{n}} \dot{x}}{n}$; $\therefore$ the given fluxion $= \dfrac{\frac{1}{n} \times x^r \dot{x}}{1 + 2lx + x^2}$, where $r = \dfrac{m - n + 1}{n}$; let $\dfrac{1}{1 + 2lx + x^2}$ be resolved, as in Fluent 94, into $\dfrac{A}{x - p}$ and $\dfrac{B}{x - q}$; then the original fluxion is transformed into two others, whose fluents can be found.

Fluent 115.　　(*De Moivre* 16.)

(334.) To find the fluent of $\dfrac{z^m \dot{z}}{1 \pm 2lz^n + z^{2n}}$, where $l = 1$; or given the fluent of $\dfrac{z^m \dot{z}}{1 \pm z^n} = p$, to find the fluent of $\dfrac{z^m \dot{z}}{(1 \pm z^n)^2}$.

Divide the numerator and denominator by z^{2n}; then

the fluxion $= \dfrac{z^{m-2n}\dot{z}}{\left(z^{-n}\pm 1\right)^2} = \dfrac{z^{-n-1}\dot{z}}{\left(z^{-n}\pm 1\right)^2} \times z^{m-n+1}$.

To find the fluent of this quantity, let $\dfrac{z^{-n-1}\dot{z}}{\left(z^{-n}\pm 1\right)^2} = \dot{x}$,

and $z^{m-n+1}=y$; then, since the fluent of $y\dot{x}=yx-f.\,x\dot{y}$,

the fluent $= \dfrac{1}{n\,.\,(z^{-n}\pm 1)} \times z^{m-n+1} -f.\,\dfrac{m-n+1}{n} \times$

$\dfrac{z^{m-n}\dot{z}}{z^{-n}\pm 1} = \dfrac{z^{m-n+1}}{n\,.\,(z^{-n}\pm 1)} - \dfrac{m-n+1}{n} \times f.\,\dfrac{z^m\dot{z}}{1\pm z^n} =$

$\dfrac{z^{m+1}}{n\,.\,(1\pm z^n)} - \dfrac{m-n+1}{n} \times p.$

SECTION VI.

(335.) *On the Fluxions of Quantities which have a variable Index ; and on the Fluents of Quantities which involve Quantities and their Logarithms, Arcs and their Sines, &c.*

EXAMPLES.

(336.) Ex. 1. To find the fluxion of x^y.

Assume $x^y = v$; then $y \times$ hyp. log. $x =$ hyp. log. v;

$\therefore \dot{y} \times$ hyp. log. $x + \dfrac{y\dot{x}}{x} = \dfrac{\dot{v}}{v}$; $\therefore \dot{v}=v\dot{y} \times$ hyp. log. x

$+ \dfrac{v y\dot{x}}{x} = x^y\dot{y} \times$ hyp. log. $x + yx^{y-1}\dot{x}$.

Cor. If x be constant, $\dot{x} = 0$; and the fluxion $=$ $x^y y \times$ hyp. log. x.

(337.) Ex. 2. The fluxion of $(a^2+z^2)^z = (a^2+z^2)^z \times$ $\dot{z} \times$ hyp. log. $(a^2 + z^2) + z\,.\,(a^2 + z^2)^{z-1}\,.\,2z\dot{z}$. This

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appears by substituting for x and its fluxion, in the last Case, the quantity $a^2 + z^2$ and its fluxion.

(338.) Ex. 3. To find the fluxion of x^{y^z}.

Let $x^y = v$; then $x^{y^z} = v^z$; assume $v^z = w$; $\therefore$ by Art. 336. $\dot{w} = v^z \dot{z} \times$ hyp. log. $v + zv^{z-1}\dot{v}$; but $v = x^y$; $\therefore$ substitute for v and its fluxion, as found in Art. 336., and we have $\dot{w} = x^{y^z} \dot{z} \times$ hyp. log. $x^y + zx^{y^{z-1}} \times (x^y \dot{y} \times$ hyp. log. $x + yx^{y-1}\dot{x}) = x^{y^z} \dot{z} \times$ hyp. log. $x^y + zx^{y^{z-1}} x^y \dot{y} \times$ hyp. log. $x + yzx^{y^{z-1}} x^{y-1}\dot{x}$.

The quantity x^{y^z} means the z power of x^y. If it denoted the y^z power of x, we must have assumed the index $y^z = v$.

In the same manner the fluxion may be found for any number of quantities.

FLUENT 116.

(339.) To find the fluent of $zy\dot{y}$, where z is a circular arc of radius 1, and sine y.

The fluent $= \dfrac{zy^2}{2} - \int . \dfrac{y^2}{2} \times \dot{z} = \dfrac{zy^2}{2} - \int . \dfrac{y^2}{2} \times$

$\dfrac{\dot{y}}{(1-y^2)^{\frac{1}{2}}} = \dfrac{zy^2}{2} - \frac{1}{4} \times$ a circular arc of radius 1, and sine

$y + \dfrac{y \cdot (1-y^2)^{\frac{1}{2}}}{4}$. (Fluent 15.)

FLUENT 117.

(340.) To find the fluent of $\dfrac{zy\dot{y}}{(1-y^2)^{\frac{1}{2}}}$, where z is the arc of a circle, and y the sine.

Let $\dfrac{y\dot{y}}{(1-y^2)^{\frac{1}{2}}} = \dot{v}$; then the given fluxion $= z\dot{v}$,

and the fluent $= zv - f.\ v\dot{z} = -z.(1-y^2)^{\frac{1}{2}} +$

$f.\ (1-y^2)^{\frac{1}{2}}.\dot{z}.$ Now $\dot{z} = \dfrac{\dot{y}}{(1-y^2)^{\frac{1}{2}}}$ to radius 1;

$\therefore (1-y^2)^{\frac{1}{2}}.\dot{z} = \dot{y}$; and the whole fluent $= -z$.

$(1-y^2)^{\frac{1}{2}} + y.$

FLUENT 118.

(341.) To find the fluent of $\dfrac{z\dot{t}}{(1+t^2)^{\frac{3}{2}}}$, where z is

a circular arc of radius 1, and tangent t.

Here $\dot{z} = \dfrac{\dot{t}}{1+t^2}$; hence, by the Rule, p. 246,

$f.\ \dfrac{z\dot{t}}{(1+t^2)^3} = \dfrac{zt}{(1+t^2)^{\frac{1}{2}}}$ (Fluent 40.) $- \dfrac{t}{(1+t^2)^{\frac{1}{2}}} \times \dfrac{\dot{t}}{1+t^2} =$

$\dfrac{zt}{(1+t^2)^{\frac{1}{2}}} - f.\ \dfrac{t\dot{t}}{(1+t^2)^{\frac{3}{2}}} = \dfrac{zt}{(1+t^2)^{\frac{1}{2}}} + \dfrac{1}{(1+t^2)^{\frac{1}{2}}}.$

(342.) The fluent of $\dfrac{z\dot{x}}{(1-x^2)^{\frac{3}{2}}}$, where z is a circular

arc of radius 1, and sine x, by a similar process, $=$

$\dfrac{zx}{(1-x^2)^{\frac{1}{2}}} - f.\ \dfrac{x\dot{x}}{1-x^2} = \dfrac{zx}{(1-x^2)^{\frac{1}{2}}} + $ hyp. log. $(1-x^2)^{\frac{1}{2}}.$

FLUENT 119.

(343.) To find the fluent of $\dfrac{z\dot{x}}{(2x-x^2)^{\frac{3}{2}}}$, where $z = $

a circular arc of radius 1, and versed sine x.

Here $\dot{z} = \dfrac{\dot{x}}{(2x-x^2)^{\frac{1}{2}}}.$ Let $\dfrac{\dot{x}}{(2x-x^2)^{\frac{3}{2}}} = \dot{y}$; then $z\dot{y}$

$= \dfrac{z\dot{x}}{(2x-x^2)^{\frac{3}{2}}}$; and $f.\ z\dot{y} = zy - f.\ y\dot{z} = \dfrac{z.(x-1)}{(2x-x^2)^{\frac{1}{2}}}$

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(Fluent 71.) $-f. \dfrac{x-1}{(2x-x^2)^{\frac{1}{2}}} \times \dfrac{\dot{x}}{(2x-x^2)^{\frac{1}{2}}} = \dfrac{z.(x-1)}{(2x-x^2)^{\frac{1}{2}}}$

$-f. \dfrac{x\dot{x}-\dot{x}}{2x-x^2} = \dfrac{z.(x-1)}{(2x-x)^{\frac{1}{2}}} - f.-1 \times \dfrac{\dot{x}-x\dot{x}}{2x-x^2} = \dfrac{z.(x-1)}{(2x-x^2)^{\frac{1}{2}}}$

$+ \text{ hyp. log. } (2x-x^2)^{\frac{1}{2}}.$

FLUENT 120.

(344.) To find the fluent of $zy^n\dot{y}$, where z is the arc of a circle, and y the sine.

This fluent $= \dfrac{zy^{n+1}}{n+1} - f. \dfrac{y^{n+1}}{n+1} \times \dot{z}$; but $\dot{z} = \dfrac{r\dot{y}}{(r^2-y^2)^{\frac{1}{2}}}$

$= \dot{y} . \left(1 - \dfrac{y^2}{r^2}\right)^{-\frac{1}{2}}$ = (by the binomial theorem) $\dot{y} \times \left(1 + \dfrac{y^2}{2r^2} + \dfrac{3y^4}{8r^4} + \&c.\right)$; $\therefore f. \dfrac{y^{n+1}\dot{z}}{n+1} = \dfrac{1}{n+1} \times \left(y^{n+1}\dot{y} + \dfrac{y^{n+3}\dot{y}}{2r^2} + \dfrac{3y^{n+5}\dot{y}}{8r^4} + \&c.\right)$ And the whole fluent $= \dfrac{zy^{n+1}}{n+1} - \dfrac{1}{n+1} \times \left(\dfrac{y^{n+2}}{n+2} + \dfrac{y^{n+4}}{(n+4).2r^2} + \dfrac{3y^{n+6}}{(n+6).8r^4} + \&c.\right)$

FLUENT 121.

(345.) To find the fluent of $z^n\dot{y}$, where $z=$ the arc of a circle, and y the corresponding sine.

Let $r=$ the radius, and x the cosine; then, by Art. 44, $r\dot{y} = x\dot{z}$, and $y\dot{z} = -r\dot{x}$.

Assume $v = z^n$; then, since the fluent of $v\dot{y} = vy - f. y\dot{v}$, we have the fluent of $z^n\dot{y} = z^n y - f. y \times nz^{n-1}\dot{z}$; but $y\dot{z} = -r\dot{x}$; $\therefore$ the second part $= + f. nrz^{n-1}\dot{x}$; and the fluent of this part $= nz^{n-1}rx - f. n.(n-1).rx z^{n-2}\dot{z}$; and since $x\dot{z} = r\dot{y}$, the fluent of the last term $= -n.(n-1).z^{n-2}r^2y + \&c.$ Hence the required fluent $= z^n y + nz^{n-1}rx - n.(n-1).z^{n-2}r^2y - \&c.$

(346.) The same fluent may be found in the following manner :

Assume $y \times (az^n + bz^{n-1} + cz^{n-2} + \&c.) = f. z^n \dot{y}$;

$$\left.\begin{array}{l} \text{then } \dot{y} \times (az^n + bz^{n-1} + cz^{n-2} + \&c.) \\ \quad + nayz^{n-1}\dot{z} + (n-1).byz^{n-2}\dot{z} + \&c. \end{array}\right\} = z^n\dot{y}.$$

Multiply $\dot{y}$ into each term of the first series, transpose $z^n\dot{y}$, and equate the corresponding coefficients; then $a\dot{y} - \dot{y} = 0$, $b\dot{y} + nay\dot{z} = 0$, $c\dot{y} + (n-1).by\dot{z} = 0$, &c. hence $a = 1$, $b\dot{y} = -nay\dot{z} = -ny\dot{z} = nr\dot{x}$; $\therefore by = nrx$,

and $b = \dfrac{nrx}{y}$, $c\dot{y} = -(n-1).by\dot{z} = -n.(n-1).\dfrac{rx}{y} \times$

$y\dot{z} = -n.(n-1).r^2\dot{y}$; $\therefore c = -n.(n-1).r^2$, &c.; and

the fluent $= y.\left(z^n + nz^{n-1}\dfrac{rx}{y} - n.(n-1).z^{n-2}r^2 - \&c.\right)$

$= z^n y + nz^{n-1}rx - n.(n-1).z^{n-2}r^2y - \&c.$ as before.

FLUENT 122.

(347.) To find the fluent of $z^n\dot{w}$, where w is the versed sine, corresponding to the arc z.

If x be the cosine, $z^n\dot{w} = -z^n\dot{x}$. To find the fluent of $-z^n\dot{x}$, assume $x \times (az^n + bz^{n-1} + cz^{n-2} + \&c.) =$ the fluent; take the fluxion, and proceed as before; the required fluent will be $= -xz^n + nrz^{n-1}y + n.(n-1) \times r^2z^{n-2}x - \&c.$

FLUENT 123.

(348.) To find the fluent of $zt^n\dot{t}$, where z is a circular arc of radius r, and tangent t.

The fluent, by Sect. II., $= \dfrac{zt^{n+1}}{n+1} - f.\dfrac{t^{n+1}}{n+1} \times \dot{z}$.

Now $\dot{z} = \dfrac{r^2\dot{t}}{r^2 + t^2}$ (Art. 44.); therefore the second term

of the fluent $= -f.\dfrac{r^2}{n+1} \times \dfrac{t^{n+1}\dot{t}}{r^2 + t^2}$.

This admits of two cases:

1. Suppose n to be an odd number; divide the numerator of this fraction by the denominator in an inverted order, as in Fluents 6, and 8, the result is

$$- \frac{r^2}{n+1} \times \left(t^{n-1}\dot{t} - r^2 t^{n-3}\dot{t} + r^4 t^{n-5}\dot{t} - \&c. \pm \text{ the} \right.$$

remainder $\left. \frac{r^{n+1}\dot{t}}{r^2 + t^2} \right)$; the number of terms in the quotient is

$\frac{n+1}{2}$, and the remainder is positive or negative according

as $\frac{n+1}{2}$ is even or odd. Hence the whole fluent $=$

$\dfrac{z t^{n+1}}{n+1} - \dfrac{r^2}{n+1} : \dfrac{t^n}{n} - \dfrac{r^2 t^{n-2}}{n-2} + \&c. \pm r^{n-1} \times$ a circular arc of radius r, and tangent t.

2. Let n be an even number; the number of terms in the quotient will be $\frac{n}{2}$, and the remainder $\pm f . \dfrac{r^n t \dot{t}}{r^2 + t^2}$,

the sign of the remainder being $+$ or $-$, according as $\frac{n}{2}$

is even or odd. The rest of the fluent is the same as

before, and the $f . \dfrac{r^n t \dot{t}}{r^2 + t^2} = r^n \times$ hyp. log. $(r^2 + t^2)^{\frac{1}{2}}$.

FLUENT 124.

(349.) To find the fluent of $v x \dot{x}$, where $v = $ hyp. log. $\left(x + (x^2 + a^2)^{\frac{1}{2}} \right)$.

In this case $\dot{v} = \dfrac{\dot{x}}{(x^2 + a^2)^{\frac{1}{2}}}$; and the fluent of $v x \dot{x}$,

$= \dfrac{v x^2}{2} - f . \dfrac{x^2}{2} \times \dot{v}$, or $- f . \dfrac{\frac{1}{2} x^2 \dot{x}}{(x^2 + a^2)^{\frac{1}{2}}}$; hence, by Fl. 14,

the whole fluent of $v x \dot{x} = \dfrac{v x^2}{2} - \frac{1}{4} \times x . (x^2 + a^2)^{\frac{1}{2}} +$

$\frac{1}{4} a^2 \times$ hyp. log. $\left(x + (x^2 + a^2)^{\frac{1}{2}} \right)$.

Fluent 125.

(350.) To find the fluent of $z^n \dot{x}$, where $z = $ hyp. log. x.

This $= z^n x - f. \dot{x} \times n z^{n-1} \dot{z}$; or since $\dot{z} = \dfrac{\dot{x}}{x}$, it $= z^n x -$

$f.\ nz^{n-1} \dot{x} = z^n x - nz^{n-1}x + n.(n-1).z^{n-2}x - \&c.$, where the law of continuation is manifest.

Fluent 126.

(351.) To find the fluent of $\dfrac{z\dot{x}}{(2x+x^2)^{\frac{3}{2}}}$, where z is

the hyp. log. of $\left(1 + x + (2x + x^2)^{\frac{1}{2}}\right)$.

Here $\dot{z} = \dfrac{\dot{x}}{(2x+x^2)^{\frac{1}{2}}}$; let $\dot{y} = \dfrac{\dot{x}}{(2x+x^2)^{\frac{3}{2}}}$; then $y =$

$-\dfrac{x+1}{(2x+x^2)^{\frac{1}{2}}}$; $\therefore f.\ z\dot{y} = zy - f.y\dot{z} = -\dfrac{zx+z}{(2x+x^2)^{\frac{1}{2}}} -$

$f.\ -\dfrac{x+1}{(2x+x^2)^{\frac{1}{2}}} \times \dfrac{\dot{x}}{(2x+x^2)^{\frac{1}{2}}} = -\dfrac{zx+z}{(2x+x^2)^{\frac{1}{2}}} + f.\dfrac{\dot{x}+x\dot{x}}{2x+x^2}$

$= -z \times \dfrac{x+1}{(2x+x^2)^{\frac{1}{2}}} + $ hyp. log. $(2x+x^2)^{\frac{1}{2}}$.

Fluent 127.

(352.) To find the fluent of $\dfrac{z\dot{x}}{(1+x^2)^{\frac{3}{2}}}$, where $z =$

hyp. log. $\left(x + (1+x^2)^{\frac{1}{2}}\right)$.

Here $\dot{z} = \dfrac{\dot{x}}{(1+x^2)^{\frac{1}{2}}}$; also if $\dfrac{\dot{x}}{(1+x^2)^{\frac{3}{2}}} = \dot{y}$, then $y =$

$\dfrac{x}{(1+x^2)^{\frac{1}{2}}}$ (Fluent 40.); $\therefore$ the fluent $= \dfrac{zx}{(1+x^2)^{\frac{1}{2}}} -$

$$f. \frac{x}{(1+x^2)^{\frac{1}{2}}} \times \frac{\dot{x}}{(1+x^2)^{\frac{1}{2}}} = \frac{z\,x}{(1+x^2)^{\frac{1}{2}}} - f. \frac{x\,\dot{x}}{1+x^2} = \frac{z\,x}{(1+x^2)^{\frac{1}{2}}} -$$

hyp. log. $(1+x^2)^{\frac{1}{2}}$.

FLUENT 128.

(353.) To find the fluent of $\dfrac{z\,\dot{x}}{(1-x^2)^{\frac{3}{2}}}$, where $z =$

hyp. log. $\dfrac{1+x}{1-x}$.

Here $\dot{z} = \dfrac{2\dot{x}}{1-x^2}$; and if $\dot{y} = \dfrac{\dot{x}}{(1-x^2)^{\frac{3}{2}}}$, $y = \dfrac{x}{(1-x^2)^{\frac{1}{2}}}$

(Fluent 40.); $\therefore f. z\dot{y} = \dfrac{z\,x}{(1-x^2)^{\frac{1}{2}}} - f. \dfrac{2\,x\,\dot{x}}{(1-x^2)^{\frac{3}{2}}} = \dfrac{z\,x}{(1-x^2)^{\frac{1}{2}}} -$

$$\frac{2}{(1-x^2)^{\frac{1}{2}}} = \frac{z\,x - 2}{(1-x^2)^{\frac{1}{2}}}.$$

FLUENT 129.

(354.) To find the fluent of $\dfrac{\dot{x}}{1-x} \times$ hyp. log. x.

By Sect. II. fluent $=$ hyp. log. $\dfrac{1}{1-x} \times$ hyp. log. $x -$

$f. \dfrac{\dot{x}}{x} \times$ hyp. log. $\dfrac{1}{1-x}$. Now since the fluxion of the

hyp. log. of $\dfrac{1}{1-x} = \dfrac{\dot{x}}{1-x} = \dot{x} + x\dot{x} + x^2\dot{x} + \&c.$; the hyp.

log. $\dfrac{1}{1-x} = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \&c.$; $\therefore f. \dfrac{\dot{x}}{x} \times$ hyp. log. $\dfrac{1}{1-x} =$

$f. \left(\dot{x} + \dfrac{x\dot{x}}{2} + \dfrac{x^2\dot{x}}{3} + \&c. \right)$; hence the whole fluent required

$=$ hyp. log. $\dfrac{1}{1-x} \times$ hyp. log. $x - x - \dfrac{x^2}{4} - \dfrac{x^3}{9} - \&c.$

FLUENT 130.

(355.) To find the fluent of $\dfrac{\dot{x}}{\left(\text{hyp. log. } \dfrac{x}{a}\right)^{\frac{1}{2}}}$.

Let hyp. log. of $\dfrac{x}{a} = v$; then if e be the number, whose hyp. log. is unity, $\dfrac{x}{a} = e^{v}$.

Also $x = a e^{v}$; and $\dfrac{\dot{x}}{x} = \dot{v}$; hence $\dfrac{\dot{x}}{\left(\text{hyp. log. } \dfrac{x}{a}\right)^{\frac{1}{2}}} = \dfrac{a e^{v} \dot{v}}{v^{\frac{1}{2}}}$

Now by Cor. to Art. 147, $e^{v} = 1 + v + \dfrac{v^{2}}{2} + \dfrac{v^{3}}{2 \cdot 3} + \&c.$;

$\therefore \dfrac{a e^{v} \dot{v}}{v^{\frac{1}{2}}} = a v^{-\frac{1}{2}} \dot{v} + a v^{\frac{1}{2}} \dot{v} + \dfrac{a v^{\frac{3}{2}} \dot{v}}{2} + \dfrac{a v^{\frac{5}{2}} \dot{v}}{2 \cdot 3} + \&c.$; $\therefore$ the

fluent required $= 2 a v^{\frac{1}{2}} + \dfrac{2 a v^{\frac{3}{2}}}{3} + \dfrac{2 a v^{\frac{5}{2}}}{5 \cdot 2} + \&c.$

FLUENT 131.

(356.) To find the fluent of $v x^{n} \dot{x}$, where $v = \text{hyp. log. } \dfrac{1}{x}$.

Here $\dot{v} = -\dfrac{\dot{x}}{x}$. Also if $x^{n} \dot{x} = \dot{w}$, since the $f. v \dot{w} = v w -$

$f. w \dot{v}$, we have the $f. v x^{n} \dot{x} = \dfrac{v x^{n+1}}{n+1} - f. \dfrac{x^{n+1}}{n+1} \times \dot{v}$. Now

$\dot{v} = -\dfrac{\dot{x}}{x}$; $\therefore$ the last term $= + f. \dfrac{x^{n+1} \times \dot{x}}{(n+1) . x} = f. \dfrac{x^{n} \dot{x}}{n+1} =$

$\dfrac{x^{n+1}}{(n+1)^{2}}$; hence the whole $= \dfrac{v x^{n+1}}{n+1} + \dfrac{x^{n+1}}{(n+1)^{2}}$.

FLUENT 132.

(357.) To find the fluent of $vx^n \dot{x}$, where $v =$ hyp. log. $(1-x.)$

This, by the method of Sect. II., $= \dfrac{v x^{n+1}}{n+1} - f \cdot \dfrac{x^{n+1} \dot{v}}{n+1}$.

Now $\dot{v} = \dfrac{-\dot{x}}{1-x}$; $\therefore$ the second term $= + \dfrac{1}{n+1} \times f \cdot \dfrac{x^{n+1} \dot{x}}{1-x}$;

or, by actual division, $= \dfrac{1}{n+1} \times \left(-x^n \dot{x} - x^{n-1}\dot{x} - \&c. \right.$

to $n+1$ terms, $\left. + \dfrac{\dot{x}}{1-x} \right)$; $\therefore$ the whole fluent $= \dfrac{v x^{n+1}}{n+1} -$

$\dfrac{1}{n+1} \times \left(\dfrac{x^{n+1}}{n+1} + \dfrac{x^n}{n} + \&c. \text{ to } n+1 \text{ terms, } + \text{ hyp. log.} \right.$

$\left. (1 - x) \right)$.

FLUENT 133.

(358.) By the same process the fluent of $vx^n \dot{x}$, where

$v =$ hyp. log. $\dfrac{1}{1-x} = \dfrac{v x^{n+1}}{n+1} + \dfrac{1}{n+1} \times \left(\dfrac{x^{n+1}}{n+1} + \dfrac{x^n}{n} + \&c. \right.$

to $n+1$ terms, $-$ hyp. log. $\left. \dfrac{1}{1-x} \right)$.

FLUENT 134.

(359.) And if $v = \frac{1}{2}$ hyp. log. $\dfrac{1+x}{1-x}$, the fluent of

$vx^n \dot{x} = \dfrac{v x^{n+1}}{n+1} + \dfrac{1}{n+1} \times \left(\dfrac{x^n}{n} + \dfrac{x^{n-2}}{n-2} + \dfrac{x^{n-4}}{n-4} + \&c. + \frac{1}{2} \right.$

hyp. log. $\left. (1 - x^2) \right)$, if n be an even number; or if n be odd, the last term within the vinculum will be $- \frac{1}{2}$ hyp. log. $\dfrac{1+x}{1-x}$.

If n be an even number, $n+1$ is odd, and the number

of terms in the quotient is $\frac{n}{2}$ (Fluent 8.) If n be odd,

$n + 1$ is even, and the number of terms is $\frac{n+1}{2}$ (Fluent 6).

(360.) To find the fluent of $z^m x^{n-1} \dot{x}$, where $z = $ hyp. log. x.

Assume the fluent $= x^n \times (az^m + bz^{m-1} + cz^{m-2} + \&c.)$;

then $nx^{n-1} \dot{x} \times (az^m + bz^{m-1} + cz^{m-2} + \&c.)$
$+ max^n z^{m-1} \dot{z} + (m-1).bx^n z^{m-2} \dot{z} + \&c.$ $\Big\} = z^m x^{n-1} \dot{x}.$

Transpose $z^m x^{n-1} \dot{x}$, and equate the corresponding terms; then $na\dot{x} - \dot{x} = 0$; $nb\dot{x} + max\dot{z} = 0$; $nc\dot{x} + (m-1)$

$bx\dot{z} = 0$, \&c.; $\therefore a = \frac{1}{n}$; $b = -\frac{m}{n^2}$; for $\dot{z} = \frac{\dot{x}}{x}$; $c = \frac{m.(m-1)}{n^3}$,

\&c. Hence the fluent $= x^n \times \Big(\dfrac{z^m}{n} - \dfrac{mz^{m-1}}{n^2} + \dfrac{m.(m-1).z^{m-2}}{n^3} - $

\&c.$\Big)$, and the law of continuation is evident.

(361.) By the same process the fluent of $z^m x^{n-1} \dot{x}$, where $z = $ hyp. log. $- x$, may be found; the result is the same.

(362.) If in Fluent 135. $m = 2$, and $n = 1$, the fluxion becomes $z^2 \dot{x}$, where $z = $ hyp. log. x. This is found immediately by Sect. II. $= z^2 x - f. 2xz\dot{z} = z^2 x - f.2z\dot{x}$,

since $\dot{z} = \dfrac{\dot{x}}{x} = z^2 x - 2zx + f. 2x\dot{z} = z^2 x - 2zx + 2x$.

(363.) To find the fluent of $Q^r x^n \dot{x}$.

Assume the fluent $= Q^r.(ax^n + bx^{n-1} + cx^{n-2} + \&c.)$,

where a, b, c, &c. are constant; then, if $m=$hyp. log. Q, we have (by Art 336.)

$$\left.\begin{array}{l} mQ^x\dot{x} \times (ax^n + bx^{n-1} + cx^{n-2}+\&c.) \\ Q^x\dot{x} \times (nax^{n-1}+(n-1).b.x^{n-2}+\&c.) \end{array}\right\} = Q^x x^n \dot{x};$$

∴ if both sides of the equation be divided by $Q^x\dot{x}$, and x^n be transposed,

$$\left.\begin{array}{l} max^n + mbx^{n-1} + mcx^{n-2}+\&c. \\ -x^n + nax^{n-1}+(n-1).b\,x^{n-2}+\&c. \end{array}\right\} =0.$$

Equate the coefficients of the corresponding terms; then $ma-1=0$, $mb+na=0$, $mc+(n-1).b=0$; hence

$$a=\frac{1}{m}, \; b=-\frac{na}{m}=-\frac{n}{m^2}, \; c=-\frac{(n-1).b}{m}=\frac{n.(n-1)}{m^3}, \; \&c.$$

∴ the fluent $= Q^x \times \left(\dfrac{x^n}{m} - \dfrac{nx^{n-1}}{m^2} + \dfrac{n.\overline{n-1}.x^{n-2}}{m^3} - \&c.,\right)$

and the law of continuation is evident.

The same result would be obtained by assuming $Q^x\dot{x}=\dot{y}$, and $x^n=z$, and finding the fluent by the Rule in Sect. II.

CHAP. XXII.

ON FLUXIONAL EQUATIONS.

(364.) THE object of this Chapter is, to shew the method of deducing from a fluxional equation the equation of the fluents, which may be called the primitive equation.

The fluxional equation, which expresses the value of $\frac{\dot{y}}{\dot{x}}$, deduced from a primitive, involving y and x, is an equation of the first order; from this may be derived another, involving $\frac{\ddot{y}}{\dot{x}^2}$, this is of the second order; and a similar mode of definition may be used in the successive fluxions.

(365.) An equation is said to be homogeneous, when the sum of the exponents of the variable quantities x and y is the same in each term; as in the equation,

$$a x \dot{x} + b y \dot{x} + d x \dot{y} + c y \dot{y} = 0,$$
$$\text{or } (ax + by) . \dot{x} + (dx + cy) . \dot{y} = 0.$$

(366.) The simplest case of fluxional equations is that in which the equation contains only one of the variable quantities with its fluxion in each term; here the fluent of each term must be taken separately.

EXAMPLES.

Ex. 1.   Let $a x^2 \dot{x} = y^3 \dot{y}$; then $\dfrac{a x^3}{3} = \dfrac{y^4}{4}$.

Ex. 2.   Let $x^n y^m \dot{x} = a\dot{y}$, or $x^n \dot{x} = a y^{-m} \dot{y}$; then $\dfrac{x^{n+1}}{n+1} =$

$\dfrac{a y^{1-m}}{1-m}$.

Ex. 3. $\dot{x} = \dfrac{2\,b^2 c\dot{y}}{(ay^3)^{\frac{1}{2}}} + \dfrac{3\,y^2\,\dot{y}}{a+b} + \dot{y}.(by+cy)^{\frac{1}{2}}$; that is, $\dot{x} =$

$\dfrac{2\,b^2 c}{a^{\frac{1}{2}}} \times y^{-\frac{3}{2}}\dot{y} + \dfrac{3\,y^2\,\dot{y}}{a+b} + y^{\frac{1}{2}}\,\dot{y}.(b+c)^{\frac{1}{2}}$; $\therefore x = - \dfrac{4\,b^2 c}{(ay)^{\frac{1}{2}}} +$

$\dfrac{y^3}{a+b} + (b+c)^{\frac{1}{2}}.\dfrac{2 y^{\frac{3}{2}}}{3}$.

Ex. 4.   Given $\dfrac{\dot{x}}{(2ax-x^2)^{\frac{1}{2}}} = \dfrac{\dot{y}}{(b^2-y^2)^{\frac{1}{2}}}$; to find the nature

of the curve, x and y being supposed to vanish together.

Here $\dfrac{1}{a} \times \dfrac{a\dot{x}}{(2ax-x)^{\frac{1}{2}}} = \dfrac{1}{b} \times \dfrac{b\dot{y}}{(b^2-y^2)^{\frac{1}{2}}}$; $\therefore$ by taking the

fluents, $\dfrac{1}{a} \times$ a circular arc of radius a, and versed sine $x =$

$\dfrac{1}{b} \times$ an arc of radius b, and sine y; $\therefore$ an arc,

whose radius is 1, and versed sine $\dfrac{x}{a}$, $=$ an arc whose

radius is 1, and sine $\dfrac{y}{b}$. But since these radii are equal,

and their arcs equal, the sines must be equal. Now the

sine of an arc, whose radius is 1, and versed sine $\dfrac{x}{a}$, $=$

$\dfrac{(2ax-x^2)^{\frac{1}{2}}}{a}$; $\therefore \dfrac{(2ax-x^2)^{\frac{1}{2}}}{a} = \dfrac{y}{b}$; and $y = \dfrac{b}{a}.(2ax-x^2)^{\frac{1}{2}}$, an

equation to the ellipse.

(367.)   The preceding rules will sometimes apply

when two or more variable quantities are involved in the equation.

Ex. 5. If $\dot{v} = y^2z^3\dot{x} + 2xz^3y\dot{y} + 3xy^2z^2\dot{z}$, then $v = xy^2z^3$.

Ex. 6. If $\dot{v} = \dfrac{\dot{x}}{y} - \dfrac{x\dot{y}}{y^2}$, then by reducing the two terms on the right to a common denominator, $\dot{v} = \dfrac{y\dot{x} - x\dot{y}}{y^2}$, and $v = \dfrac{x}{y}$.

(368.) In some cases the quantities and their fluxions are so involved, that these methods cannot be applied. The equations of this class are not reducible to any general form; but the variable quantities may frequently be separated by some of the following Rules.

(369.) RULE 1. Substitute for the sum of the two variable quantities, for the sum of their squares, for their products, &c.

Ex. 7. Let $a^2\dot{x} = (x + y)^2 \times \dot{y}$.

Assume $x + y = z$; then $\dot{x} + \dot{y} = \dot{z}$, and $\dot{x} = \dot{z} - \dot{y}$; $\therefore$ by substitution, $a^2\dot{z} - a^2\dot{y} = z^2\dot{y}$, and $\dot{y} = \dfrac{a^2\dot{z}}{a^2 + z^2}$, whose fluent is known by Art. 44.

Ex. 8. Let $(x\dot{y} + y\dot{x}) \cdot (a^4 - x^2y^2)^{\frac{1}{2}} = \dfrac{x\dot{x} + y\dot{y}}{(x^2 + y^2)^{\frac{3}{4}}}$.

Assume $xy = z$, and $x^2 + y^2 = v$; then $x\dot{y} + y\dot{x} = \dot{z}$, and $x\dot{x} + y\dot{y} = \dfrac{\dot{v}}{2}$; $\therefore \dot{z}.(a^4 - z^2)^{\frac{1}{2}} = \dfrac{\dot{v}}{2v^{\frac{3}{4}}}$, the fluents of which are known.

(370.) RULE 2. Multiply or divide the given equation by some function of the unknown quantities, so as to bring it to a form which is known.

Ex. 9. Let $my\dot{x} + ny\dot{x} = 0$.

Divide the terms by xy; then $\dfrac{m\dot{x}}{x} + \dfrac{n\dot{y}}{y} = 0$; $\therefore m \times$ hyp. log. $x + n \times$ hyp. log. $y =$ some given quantity $= c$.

Ex. 10. Let $\dfrac{\dot{x}}{x} + \dfrac{\dot{y}}{y} = \dfrac{ax^m \dot{x}}{y^n}$.

Multiply the equation by $nx^n y^n$; then $ny^n x^{n-1} \dot{x} + nx^n y^{n-1} \dot{y} = nax^{m+n} \dot{x}$; and by taking the fluent, $x^n y^n = \dfrac{nax^{m+n+1}}{m+n+1}$; and if no correction be wanted, $y^n = \dfrac{nax^{m+1}}{m+n+1}$.

Ex. 11. Let $\dfrac{p\dot{x}}{x} + \dfrac{r\dot{y}}{y} = \dfrac{x^m \dot{x}}{ay^r}$.

Multiply by $x^p y^r$; then $px^{p-1} \dot{x} \times y^r + x^p \times ry^{r-1} \dot{y} = \dfrac{x^{m+p} \dot{x}}{ay^o}$. The fluent of the first part is $x^p y^r$; and since $y^o = 1$, the fluent of the second part $= \dfrac{x^{m+p+1}}{(m+p+1).a}$; therefore, if there be no correction, $y^r = \dfrac{x^{m+1}}{(m+p+1).a}$.

Ex. 12. Let $\dfrac{a\dot{x}}{x} + \dfrac{b\dot{y}}{y} - \dfrac{c\dot{z}}{z} = 0$.

Multiply by $\dfrac{x^a y^b}{z^c}$; then the resulting fluxion is $\dfrac{ax^{a-1} \dot{x} y^b}{z^c} + \dfrac{by^{b-1} \dot{y} x^a}{z^c} - \dfrac{cx^a y^b \dot{z}}{z^{c+1}} = 0$; and the fluent is $\dfrac{x^a y^b}{z^c} = A$.

The same method may be applied to fluxional equations of higher orders.

Ex. 13. Let $\ddot{x} - x\dot{z}^2 = b\dot{z}^2$.

Here $\ddot{z}$ is not found ; therefore in deducing the fluxional equation, $\dot{z}$ must have been supposed constant. Multiply by $\dot{x}$; then $\dot{x}\ddot{x} - x\dot{x}\dot{z}^2 = b\dot{x}\dot{z}^2$; and as $\dot{z}$ is constant, the fluent is $\dfrac{\dot{x}^2}{2} - \dfrac{x^2\dot{z}^2}{2} = bx\dot{z}^2$; therefore $\dot{z} =$

$\dfrac{\dot{x}}{(2bx + x^2)^{\frac{1}{2}}}$, and $z = \text{hyp. log.} \left(b + x + (2bx + x^2)^{\frac{1}{2}}\right)$.

(371.) RULE 3. It is sometimes useful to substitute for one of the unknown quantities, the sum of the other, and a new variable quantity.

Ex. 14. Let $a\dot{z} = z\dot{x} - x\dot{x}$.

Assume $z = a + x + v$; then $\dot{z} = \dot{x} + \dot{v}$; ∴ by substituting for z and $\dot{z}$, we have $a\dot{x} + a\dot{v} = a\dot{x} + x\dot{x} + v\dot{x} - x\dot{x}$, or

$a\dot{v} = v\dot{x}$; ∴ $\dot{x} = \dfrac{a\dot{v}}{v}$, and $x = a \times \text{hyp. log.} \ v$. Hence

$z = a + v + a \times \text{hyp. log.} \ v$. Substitute for v its value, $z - a - x$, and $x = a \times \text{hyp. log.} \ (z - a - x)$.

(372.) RULE 4. Substitute for one fluxion or for one unknown quantity, the product of the other and some new variable quantity. This mode of substitution may be adopted in homogeneous equations.

Ex. 15. Let the equation be $\dfrac{\dot{y}}{y} - \dfrac{\dot{x}}{x} = \dfrac{x^3\dot{x}}{ay^2}$.

Assume $y = zx$; then $\dot{y} = z\dot{x} + x\dot{z}$. Hence, by substitution, $\dfrac{z\dot{x} + x\dot{z}}{zx} - \dfrac{\dot{x}}{x} = \dfrac{x^3\dot{x}}{az^2x^2}$, or $\dfrac{\dot{z}}{z} = \dfrac{x^3\dot{x}}{az^2x^2}$; ∴ $az\dot{z} = x\dot{x}$;

and if no correction be wanted, $x^2 = az^2 = \dfrac{ay^2}{x^2}$, and $x^4 = ay^2$.

Ex. 16. Let $x\dot{x} + ay\dot{x} + y\dot{y} = 0$.

Assume $y = zx$; ∴ $\dot{y} = z\dot{x} + x\dot{z}$. Hence, by substitution, $x\dot{x} + azx\dot{x} + z^2x\dot{x} + x^2z\dot{z} = 0$; or $x\dot{x}.(1 + az + z^2) = - x^2z\dot{z}$;

$\therefore -\dfrac{\dot{x}}{x}=\dfrac{z\dot{z}}{1+az+z^2}$, whose fluent is known by Fluent 101.

Ex. 17. To find the fluent of $ax\dot{x} + by\dot{x} + dx\dot{y} + ey\dot{y} = 0$.

Let $y = zx$; then $\dot{y}=z\dot{x} + x\dot{z}$; therefore by substitution, $ax\dot{x}+bzx\dot{x} + dzx\dot{x}+dx^2\dot{z}+ez^2x\dot{x}+ex^2z\dot{z}=0$. Assume $c=b+d$; then $x\dot{x}.(a+cz+ez^2) + (d\dot{z}+ez\dot{z}).x^2 = 0$, or $\dfrac{\dot{x}}{x} =-\dfrac{d\dot{z}+ez\dot{z}}{a+cz+ez^2}$.

The fluent of the first part = hyp. log. of x; and that of the second is found by Fluents 100. and 101.

Ex. 18. Let $x\dot{y} - y\dot{x} = \dot{x}.(x^2 + y^2)^{\frac{1}{2}}$.

Assume $y = zx$; then, by substituting for y and $\dot{y}$, $zx\dot{x} + x^2\dot{z} - zx\dot{x}= \dot{x}.(x^2+z^2 x^2)^{\frac{1}{2}}$, or $x^2\dot{z}=x\dot{x}.(1+z^2)^{\frac{1}{2}}$; therefore $\dfrac{\dot{z}}{(1+z^2)^{\frac{1}{2}}}=\dfrac{\dot{x}}{x}$; an hyp. log. $\left(z+(1+z^2)^{\frac{1}{2}}\right) =$ hyp. log. x.

Ex. 19. Let $\dfrac{\dot{x}}{x} - \dfrac{\dot{y}}{y} = \dfrac{x^m\dot{x}}{ay^n}$.

Assume $y=zx$; then $\dot{y}=z\dot{x} +x\dot{z}$. Hence by substitution, $\dfrac{\dot{x}}{x} - \dfrac{z\dot{x}+x\dot{z}}{zx} = \dfrac{x^m\dot{x}}{az^nx^n}$, or $-\dfrac{\dot{z}}{z} = \dfrac{x^m\dot{x}}{az^nx^n}$; that is,

$-z^{n-1}\dot{z}=\dfrac{x^{m-n}\dot{x}}{a}$; $\therefore$ cor. $-\dfrac{z^n}{n} = \dfrac{x^{m-n+1}}{a.(m-n+1)}$; but $z= \dfrac{y}{x}$; $\therefore$ by substitution, cor. $-\dfrac{y^n}{n} = \dfrac{x^{m+1}}{a.(m-n+1)}$.

(373.) RULE 5. If only one of the variable quantities be found in the equation, substitute for one fluxion the product of the other, and some new variable quantity.

Ex. 20. Let $a\dot{x}\dot{y}^3 = y.(\dot{x}^2 + \dot{y}^2)^2$.

Let $\dot{x} = z\dot{y}$; then $az\dot{y}^4 = y \cdot (z^2\dot{y}^2 + \dot{y}^2)^2 = y\dot{y}^4.(z^2 + 1)^2$;

$\therefore y = \dfrac{az}{(z^2 + 1)^2}$; hence $\dot{y} = \dfrac{a\dot{z} - 3az^2\dot{z}}{(z^2 + 1)^3}$; therefore $\dot{x}$ or $z\dot{y} =$

$\dfrac{az\dot{z} - 3az^3\dot{z}}{(z^2 + 1)^3}$, whose fluent or x, by Fluent 55, $=$

$\dfrac{3a}{2.(1 + z^2)} - \dfrac{a}{(1 + z^2)^2}$.

Ex. 21. Let $-\dfrac{\dot{y}}{\dot{x}^2 + \dot{y}^2} = \dfrac{1}{y}$.

Here $\dot{x}$ is constant. Assume $\dot{y} = z\dot{x}$. Then $\ddot{y} = \dot{z}\dot{x}$;

hence, by substitution, $-\dfrac{\dot{z}\dot{x}}{\dot{x}^2 + z^2\dot{x}^2} = \dfrac{1}{f.z\dot{x}}$; $\therefore -\dot{z}\dot{x}.f.z\dot{x}$

$= \dot{x}^2 + z^2\dot{x}^2$, or $-f.z\dot{x} = \dfrac{(1 + z^2) \cdot \dot{x}}{\dot{z}}$. Take the fluxion;

then $-z\dot{x} = 2z\dot{x} - \dfrac{(1 + z^2).\ddot{z}\dot{x}}{\dot{z}^2}$; $\therefore \dfrac{\ddot{z}}{\dot{z}} = \dfrac{3z\dot{z}}{1 + z^2}$; and

hyp. log. $\dot{z}$ + cor. $= \dfrac{3}{2} \times$ hyp. log. $(1 + z^2)$.

Since $\dot{z}$ or $\dot{x}$ must enter into every term, let $\dot{z}$ be corrected by taking it $= a\dot{x}$, a given quantity; then hyp.

log. $\dfrac{\dot{z}}{a\dot{x}} = \dfrac{3}{2} \times$ hyp. log. $(1 + z^2)$; $\therefore \dfrac{\dot{z}}{a\dot{x}} = (1 + z^2)^{\frac{3}{2}}$, and

$a\dot{x} = \dfrac{\dot{z}}{(1 + z^2)^{\frac{3}{2}}}$; $\therefore z\dot{x}$, or $\dot{y} = \dfrac{z\dot{z}}{a.(1 + z^2)^{\frac{3}{2}}}$; hence (Fluent

40.) $x = \dfrac{z}{a.(1 + z^2)^{\frac{1}{2}}}$ + cor., and $y = -\dfrac{1}{a.(1 + z^2)^{\frac{1}{2}}}$ + cor.

Ex. 22. Let $y\dot{y}^2\dot{x} = a\dot{x}^4 + 2a\dot{x}^2\dot{y}^2 + a\dot{y}^4$.

Take $z\dot{y} = \dot{x}$; then $yz\dot{y}^4 = az^4\dot{y}^4 + 2az^2\dot{y}^4 + a\dot{y}^4$;

$\therefore y = az^3 + 2az + \dfrac{a}{z}$, and $\dot{y} = 3az^2\dot{z} + 2a\dot{z} - \dfrac{a\dot{z}}{z^2}$. Now

$\dot{x} = z\dot{y}$; $\therefore \dot{x} = 3az^3\dot{z} + 2az\dot{z} - \dfrac{a\dot{z}}{z}$, and $x = \dfrac{3 \cdot az^4}{4} + az^2 - a$

$\times$ hyp. log. z.

x may be found in terms of y, by obtaining the value of

z from the equation $y = az^3 + 2az + \dfrac{a}{z}$, and substituting

it in the equation $x = \dfrac{3az^4}{4} + az^2 - a \times$ hyp. log. z.

Ex. 23.   Let $x = \dfrac{y\dot{x}}{\dot{y}} + \dfrac{ay^2\dot{x}^2}{\dot{y}^2} + \dfrac{by^3\dot{x}^3}{\dot{y}^3} + $ &c.

Assume $\dot{x} = z\dot{y}$; then, by substitution, $x = zy + az^2y^2$
$+ bz^3y^3 + $ &c.

In this equation, for zy write v; then $x = v + av^2 + bv^3$
$+ $&c.; $\therefore \dot{x}$ or $z\dot{y} = \dot{v} + 2av\dot{v} + 3bv^2\dot{v} + $&c.; that is,

$\dfrac{v\dot{y}}{y} = \dot{v} + 2av\dot{v} + 3bv^2\dot{v} + $ &c. ; $\therefore$ hyp log. $y = $ hyp. log.

$v + 2av + \dfrac{3bv^2}{2} + $ &c.

(374.) RULE 6.   Take the fluxion of the given equation, one of the fluxions being supposed constant.

Ex. 24.   Let the equation be $y + \dfrac{a\dot{y} - x\dot{y}}{\dot{x}} = x - \dfrac{y\dot{x}}{\dot{y}}$.

Make $\dot{x}$ constant, and take the fluxion; then $\dot{y} +$

$\dfrac{a\ddot{y} - x\dot{y} - \dot{y}\dot{x}}{\dot{x}} = \dot{x} + \dfrac{\ddot{y}y\dot{x} - \dot{y}^2\dot{x}}{\dot{y}^2}$; $\therefore \ddot{y} \times \dfrac{a - x}{\dot{x}} = \dfrac{\ddot{y}y\dot{x}}{\dot{y}^2}$, and

$\dfrac{\dot{y}^2}{y} = \dfrac{\dot{x}^2}{a - x}$; hence $y^{-\frac{1}{2}}\dot{y} = (a - x)^{-\frac{1}{2}} \times \dot{x}$; therefore

$2y^{\frac{1}{2}} = -2.(a - x)^{\frac{1}{2}}$.

Ex. 25.   The equation $\dfrac{a\dot{x} + y\dot{x}}{\dot{y}} = x + y - \dfrac{x\dot{y}}{\dot{x}}$, solved

in the same manner, gives $2x^{\frac{1}{2}} = 2.(a + y)^{\frac{1}{2}}$.

Ex. 26. Let $y\ddot{y}\dot{x} - x\dot{y}^2 = a\dot{x}^2$.

Take $\dot{x}$ constant; then $\dot{y}^2\dot{x} + y\ddot{y}\dot{x} - \dot{x}\dot{y}^2 - 2xy\ddot{y} = 0$, or

$y\ddot{y}\dot{x} = 2x\dot{y}\ddot{y}$; $\therefore y\dot{x} = 2\dot{y}x$, or $\dfrac{\dot{x}}{x} = \dfrac{2\dot{y}}{y}$; hence hyp. log.

$x = 2 \times$ hyp. log. y, and $ax = y^2$.

(377.) RULE 7. The equation may sometimes be brought to a better form by completing the square and extracting the root.

Ex. 27. Let $\dot{y}^2 = \dot{x}\dot{y} - x^2\dot{x}^2$.

Hence $\dot{y}^2 - \dot{x}\dot{y} + \dfrac{\dot{x}^2}{4} = \dfrac{\dot{x}^2}{4} - x^2\dot{x}^2$; $\therefore \dot{y} - \dfrac{\dot{x}}{2} = \pm\dot{x}.(\tfrac{1}{4} - x^2)^{\frac{1}{2}}$;

$\therefore \dot{y} = \dfrac{\dot{x}}{2} \pm \dot{x}\cdot(\tfrac{1}{4} - x^2)^{\frac{1}{2}}$, and $y = \dfrac{x}{2} \pm$ the area $GPEA$,

where $GP = x$, and $GE = \tfrac{1}{2}$. (Fig. p. 255.)

Ex. 28. Let the equation be $x^2\dot{x}^2 + xy\dot{x}\dot{y} = a^2\dot{y}^2$.

Complete the square; then $x^2\dot{x}^2 + x\dot{x}y\dot{y} + \dfrac{y^2\dot{y}^2}{4} = a^2\dot{y}^2$

$+ \dfrac{y^2\dot{y}^2}{4}$; therefore, by extracting the square root, $x\dot{x} +$

$\dfrac{y\dot{y}}{2} = \dot{y}.(a^2 + \tfrac{1}{4}y^2)^{\frac{1}{2}} = \dfrac{\dot{y}}{2}.(4a^2 + y^2)^{\frac{1}{2}} = \dfrac{2a^2\dot{y}}{(4a^2 + y^2)^{\frac{1}{2}}} + \dfrac{\tfrac{1}{2}y^2\dot{y}}{(4a^2 + y^2)^{\frac{1}{2}}}$;

the fluents of which are known by Art. 43. and 214.

(376.) PROB. To transform a fluxional equation of the second order, containing two variable quantities x and y, of which x flows uniformly into another in which y shall flow uniformly, or $\dot{y}$ be constant.

Let MA be a curve, whose abscissa AP flows with a uniform velocity; let $AP = x$, $Pp = \dot{x}$, $PM = y$, and $2mf = \ddot{y}$. Next, suppose y to flow uniformly; let $Mh = \dot{y}$; then $he = \dot{x}$, and $2me = \ddot{x}$. Now by similar triangles, Mhe, fme,
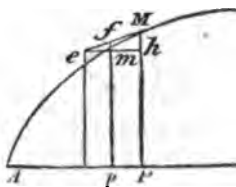
$eh\ (\dot{x}) : hM\ (\dot{y}) :: 2\,em\ (\ddot{x}) : 2\,mf\ (-\ddot{y}); \ \therefore\ \ddot{y} = -\dfrac{\dot{y}\ddot{x}}{\dot{x}}$, which, being substituted for $\ddot{y}$ in the original equation, will give another equation in which $\dot{y}$ is constant.

(377.) The same proposition may be thus demonstrated :

Let $y = a + bx + cx^2 + dx^3 +$ &c. ; then $\dfrac{\dot{y}}{\dot{x}} = b + 2cx + 3dx^2 +$ &c.

Assume $\dot{x}$ constant, and take the fluxion ; then $\dfrac{\ddot{y}}{\dot{x}} = 2c\dot{x} + 6dx\dot{x} +$ &c.

Next make $\dot{y}$ constant, and take the fluxion ; then $-\dfrac{\dot{y}\ddot{x}}{\dot{x}^2}$ $= 2cx + 6dx\dot{x} +$ &c. ; $\therefore \dfrac{\ddot{y}}{\dot{x}}$, when $\dot{x}$ is constant, $=$ $-\dfrac{\dot{y}\ddot{x}}{\dot{x}^2}$; when $\dot{y}$ is constant, or $\ddot{y}$, $= -\dfrac{\dot{y}\ddot{x}}{\dot{x}}$.

(378.) RULE 8. Any fluxional equation of the second order, containing only x and y, in which $\dot{x}$ is constant, may be transformed into another in which $\dot{y}$ is constant, by substituting for $\ddot{y}$ the fraction $-\dfrac{\dot{y}\ddot{x}}{\dot{x}}$.

Ex. 29. Let $\dot{x}\dot{y} - x\ddot{y} - a\ddot{y} - \dfrac{x\dot{y}^2}{b} = 0.$

In this case, since $\ddot{x}$ is not found, $\dot{x}$ is constant. Substitute according to the Rule for $\ddot{y}$; then $\dot{x}\dot{y} + x \times \dfrac{\dot{y}\ddot{x}}{\dot{x}} +$ $a \times \dfrac{\dot{y}\ddot{x}}{\dot{x}} - \dfrac{x\dot{y}^2}{b} = 0$, or $\dot{x}^2 + x\ddot{x} + a\ddot{x} - \dfrac{x\dot{x}\dot{y}}{b} = 0$; the fluent $= x\dot{x} + a\dot{x} - \dfrac{x^2\dot{y}}{2b}$, which must be equal to some constant

quantity. Now since $\dot{x}$ or $\dot{y}$ must be found in every term of the first fluxion of the equation, the constant quantity must be of the form $c\dot{y}$; $\therefore$ take $x\dot{x} + a\dot{x} - \dfrac{x^2\dot{y}}{2b} = c\dot{y}$; then

$$\dot{y} = \frac{2bx\dot{x}}{2bc+x^2} + \frac{2ab\dot{x}}{2bc+x^2}, \text{ and } y = b \times \text{hyp. log. } (2bc+x^2)$$

$+ \dfrac{a}{c} \times$ a circular arc, whose radius is $(2bc)^{\frac{1}{2}}$, and tangent x.

(379.) This equation may also be solved in the following manner:

Multiply it by the fraction $\dfrac{\dot{x}}{\dot{y}^2}$; the result is $\dfrac{\dot{x}^2\dot{y} - x\dot{x}\ddot{y}}{\dot{y}^3}$

$- \dfrac{a\dot{x}\ddot{y}}{\dot{y}^2} - \dfrac{x\dot{x}}{b} = 0$; and the fluent $\dfrac{x\dot{x}}{\dot{y}} + \dfrac{a\dot{x}}{\dot{y}} - \dfrac{x^2}{2b} = $ some

constant quantity $= c$; $\therefore x\dot{x} + a\dot{x} - \dfrac{x^2\dot{y}}{2b} = c\dot{y}$, the same

as in the last case.

(380.) RULE 9. For the quantity, the value of which is to be found, assume a series involving the powers of the other with unknown coefficients; substitute this series and its fluxions for their values in the given equation, and equate the coefficients of the corresponding terms.

(381.) Ex. 30. Let $\dfrac{m\dot{x}}{1+x} = \dot{y}$, or $m\dot{x} - \dot{y} - x\dot{y} = 0$.

Assume the series, and follow the process adopted in

Art. 147; $x = \dfrac{y}{m} + \dfrac{y^2}{2m^2} + $ &c.

Ex. 31. Let $y\dot{x}^2 + m\dot{x}\dot{y} - m^2\ddot{y} = 0$.

In this case $\dot{x}$ is constant. Assume $\dot{x} = 1$, and let $y = ax + bx^2 + cx^3 + dx^4 + $&c.; then $\dot{y} = a + 2bx + 3cx^2 + 4dx^3 + $ &c., and $\ddot{y} = 2b + 2.3.cx + 3.4.dx^2 + $ &c.; hence, by substitution in the original equation,

$$\left.\begin{array}{r} ax + bx^2 + \&c. \\ + ma + 2mbx + 3mcx^2 + \&c. \\ - 2m^2 b - 2\cdot 3m^2 cx - 3\cdot 4\cdot m^2 dx^2 - \&c. \end{array}\right\} = 0.$$

Hence $ma - 2m^2 b = 0$, $a + 2mb - 2.3.m^2 c = 0$, $b + 3mc -$

$3.4.m^2 d = 0$, &c.; $\therefore b = \dfrac{ma}{2m^2} = \dfrac{a}{2m}$, $c = \dfrac{2a}{2.3.m^2}$, &c.;

$\therefore y = ax + \dfrac{ax^2}{2m} + \dfrac{2ax^3}{2.3m^2} + \&c. = a \times \left(x + \dfrac{x^2}{2m} + \dfrac{2x^3}{2.3m^2} \right.$

$+ \&c. \Big)$

(382.) Ex. 32. Let $mx^2\dot{x} + y\dot{x} - ny = 0$.

Assume $y = ax^3 + bx^4 + cx^5 + \&c.$; $\therefore \dot{y} = 3ax^2\dot{x} + 4bx^3\dot{x}$
$+ 5cx^4\dot{x} + \&c.$ And by substitution in the given equation,

$$\left.\begin{array}{r} mx^2\dot{x} + ax^3\dot{x} + bx^4\dot{x} + \&c. \\ - 3nax^2\dot{x} - 4nbx^3\dot{x} - 5ncx^4\dot{x} - \&c. \end{array}\right\} = 0.$$

Hence $m - 3na = 0$, $a - 4nb = 0$, $b - 5nc = 0$, &c.;

$\therefore a = \dfrac{m}{3n}$, $b = \dfrac{a}{4n} = \dfrac{m}{3.4.n^2}$, $c = \dfrac{b}{5n} = \dfrac{m}{3.4.5n^3}$, &c.; $\therefore y =$

$\dfrac{mx^3}{3n} + \dfrac{mx^4}{3.4.n^2} + \dfrac{mx^5}{3.4.5.n^3} + \&c.$

If in this equation we had assumed $y = Ax + Bx^2 + Cx^3$
$+ \&c.$, it would have appeared that $A = 0$, and $B = 0$; so
that a regular series beginning with x is not always re-
quired. The principal difficulty is to determine, what
kind of series with respect to the indices ought to be
taken, that no superfluous terms may be admitted. As
the point is curious, rather than useful, the Reader is
referred to *Simpson's Fluxions.* See also on the subject
of this Chapter, *Agnesi's Analytical Institutions,* Vol. II.
and *Bossut's Traités de Calcul. Différentiel.*

Chap. XXIII.

ON THE MAXIMA AND MINIMA OF CURVES.

Proposition.

(383.) To find the nature of curves in which some conditions being invariable, others become the greatest or the least possible.

The method of solving problems of this kind, will appear by an Example.

Examples.

(384.) Ex. 1. Given the length of a curve, required its nature, when the area is a maximum.

It is evident, that by merely putting the fluent of $y\dot{x}$ a maximum, no solution can be obtained; for no limitation is expressed, and the fluent will admit of increase without limit. But as the length is given, the $f.\dot{x}$, so far as concerns the $f.y\dot{x}$, is a given quantity; therefore the fluent of $y\dot{x} \pm f.\dot{x}$ must also be a maximum; or, to render the terms homogeneous, that they may admit of comparison, $f.y\dot{x} \pm f.a\dot{x}$ must be a maximum.

Now, if for every individual value of y, this flowing quantity be constantly a maximum, the whole fluent will be so; but for every such individual value of y, the flowing quantity is $y\dot{x} \pm a\dot{x}$. Hence, the nature of the curve will be determined by ascertaining, what relations

of $\dot{x}$ and $\dot{z}$ will render $y\dot{x} \pm a\dot{z}$ a maximum, for any given value of y; or the fluxion of $y\dot{x} \pm a\dot{z}$ must $=0$, whilst y is constant; and this must be the case for every successive value of y throughout; so that in each limiting portion of the area, for every value of y the ratio of $\dot{x}$ and $\dot{z}$ must be such, as to make $y\dot{x} \pm a\dot{z}$ a maximum.

$\therefore y\ddot{x} \pm a\ddot{z} = 0$; but $\dot{z}^2 = \dot{y}^2 + \dot{x}^2$; $\therefore \dot{z}\ddot{z} = \dot{x}\ddot{x}$, and $\ddot{z} = \dfrac{\dot{x}\ddot{x}}{\dot{z}}$; $\therefore y\ddot{x} = \mp \dfrac{a\dot{x}\ddot{x}}{\dot{z}}$, and $y\dot{z} = \mp a\dot{x}$. But from the nature of the problem, $y\dot{z}$ must be positive; and therefore the true result is $y\dot{z} = a\dot{x}$.

Now in the circle, $a : y :: \dot{z} : \dot{x}$; $\therefore a\dot{x} = y\dot{z}$. Hence, the circle is the curve required; in which the length being given, the area is a maximum.

The same mode of reasoning may be adopted in the following cases. Hence the Rule.

(385.) If A and B denote any functions of x and y, and $\dot{x} = (c^2 \pm \dot{y}^2)^{\frac{1}{2}}$, where c is constant, the expression $A\dot{x} \pm B\dot{y}$ is a maximum or minimum, when $A\dot{y} = \mp B\dot{x}$, or the functions of x and y are reciprocal.

Ex. 2. To determine the nature of a curve line which generates the greatest solid under a given surface.

Here $f. 2py\dot{z}$ or $f. ay\dot{z}$ is given, and $f. y^2\dot{x}$ is a maximum; hence $f. ay\dot{z} \pm f. y^2\dot{x}$ is a maximum; $\therefore ay\dot{x} = y^2\dot{z}$, or $a\dot{x} = y\dot{z}$, a property of the circle, and the solid is a sphere.

This is a particular case of the problem; if more generally we take $ay\dot{x} = y^2\dot{z} - c\dot{z} = (y^2 - c).\dot{z}$;

$$\text{then } a^2y^2\dot{x}^2 = \dot{z}^2 . (y^2 - c)^2$$
$$= (\dot{x}^2 + \dot{y}^2) . (y^2 - c)^2$$

and $\dot{x} = \dfrac{\dot{y} . (y^2 - c)}{\left(a^2y^2 - (y^2 - c)^2\right)^{\frac{1}{2}}}$. In the preceding case

$c = 0$.

Ex. 3. To determine the nature of the curve, so that the length of the arc being given, the solid may be a maximum.

Here $f.\dot{z}$, or $f.a^2\dot{z}$, is given, and $f.y^2\dot{x}$ is a maximum; $\therefore f.a^2\dot{z} \pm f.y^2\dot{x}$ is a maximum; and $a^2\dot{x} = y^2\dot{z}$, the

equation, or $\dot{x} = \dfrac{y^2\dot{y}}{(a^4 - y^4)^{\frac{1}{2}}}$.

This also is a particular case of the problem.

If we take $a^2\dot{x} = (y^2 - c).\dot{z}$, as in the preceding

Example, $\dot{x} = \dfrac{(y^2 - c).\dot{y}}{\left(a^4 - (y^2 - c)^2\right)^{\frac{1}{2}}}$.

Ex. 4. To find the nature of the curve which generates the solid of least resistance, when moving in a fluid, in the direction of its axis; its greatest ordinate AB, and length DE, being given.

If $DH = x$, $HF = y$, and $FC = \dot{z}$, the resistance on

$CF \propto \dfrac{y\dot{y}^3}{\dot{z}^4}$, $\propto \dfrac{y\dot{y}^3}{\dot{z}^3} \times \dot{z}$, the fluent of

which is a minimum. Also, the

$f.a\dot{x}$ is given; $\therefore f.\dfrac{y\dot{y}^3}{\dot{z}^3} \times \dot{z} \pm$

$f.a\dot{x}$ is a minimum; hence, $\dfrac{y\dot{y}^3}{\dot{z}^3} \times \dot{x}$

$= a\dot{z}$, the equation to the curve.
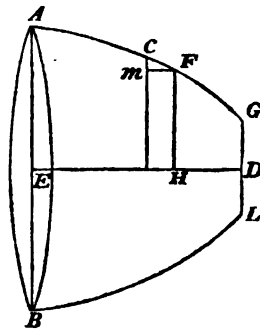
Cor. The curve cannot meet the axis, for $y = \dfrac{a \times \dot{z}^4}{\dot{y}^3\dot{x}} =$

$a \times \dfrac{CF^4}{Cm^3 \times Fm}$, where the numerator is greater than

the denominator; and therefore y is greater than a.

Ex. 5. Given the area of the generating plane, and

the greatest ordinate; to find the curve which generates the solid of least resistance.

Here $f.\,ay\dot{x}$ is given, whilst $f.\,\dfrac{y\ddot{y}^3}{\dot{z}^3}\times\dot{z}$ is a minimum;

$\therefore f.\,\dfrac{y\ddot{y}^3}{\dot{z}^3}\times\dot{z}\pm f.\,ay\dot{x}$ is a minimum; hence, $ay\dot{z}=$

$\dfrac{y\ddot{y}^3}{\dot{z}^3}\times\dot{x}$, and $a=\dfrac{\ddot{y}^3\dot{x}}{\dot{z}^4}$, the equation to the curve.
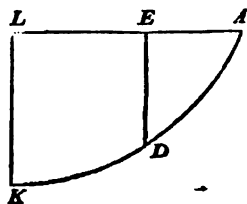
Cor. Since $\dfrac{\ddot{y}^3\dot{x}}{\dot{z}^4}$ is constant, the angle contained between the tangent FC and Fm is invariable; hence, the solid is either a cone, or the frustum of a cone.

Ex. 6. To find the same, when the greatest ordinate and content are given.

Here $f.\,\dfrac{y\ddot{y}^3}{\dot{z}^3}\times\dot{z}$ is a minimum, and $f.\,y^2\dot{x}$ is given;

or making the terms of the same dimensions, $f.\,\dfrac{ay\ddot{y}^3}{\dot{z}^3}\times\dot{z}$

$\pm f.\,y^2\dot{x}$ is a minimum; hence, $\dfrac{ay\ddot{y}^3}{\dot{z}^3}\times\dot{x}=y^2\dot{z}$; and

$a=\dfrac{y\dot{z}^4}{\ddot{y}^3\dot{x}}$, the equation to the curve. The fluent is found in Ex. 20., on Fluxional Equations.

Ex. 7. To determine the nature of a curve ADK, down which a body will descend from A to K in the shortest time possible; the points A and K being given, and the velocity varying as the mth power of the ordinate.

Since $\dot{t}\propto\dfrac{\dot{z}}{v}$, where $\dot{z}$ is the flux-

ion of the curve, if $y=DE$, $\dot{t}\propto$

$\dfrac{\dot{z}}{y^m}$. Now the points A and K are

given; therefore $f \cdot \dfrac{\dot{x}}{a^m}$ is given; and $f \cdot \dfrac{\dot{z}}{y^m} \pm f \cdot \dfrac{\dot{x}}{a^m}$ is a minimum. Hence, by the Rule, $y^m \dot{z} = a^m \dot{x}$, the equation to the curve.

COR. 1. If the velocity $\propto$ as the square root of the ordinate, $m = \dfrac{1}{2}$, and $y^{\frac{1}{2}} \dot{z} = a^{\frac{1}{2}} \dot{x}$, which is a property of the cycloid, the curve commencing at A.

For $IH : IL :: CE : ED :: CA : AE$
$\quad\quad\quad\quad :: CA^{\frac{1}{2}} : AD^{\frac{1}{2}}$

or, $\quad \dot{z} : \dot{x} :: a^{\frac{1}{2}} : y^{\frac{1}{2}}$;

$\quad \therefore y^{\frac{1}{2}} \dot{z} = a^{\frac{1}{2}} \dot{x}.$

COR. 2. If $m = 1$, $a\dot{x} = y\dot{z}$, which is the property of the circle.

EX. 8. To determine the nature of the curve, down which a body will descend from an horizontal line AL to a vertical line LK, (See Fig. in the preceding page,) in the least time possible; the area being given, and the velocity varying as the mth power of the ordinate.

Here the $f.$ of $y\dot{x}$ is given; and since $\dot{t} \propto \dfrac{\dot{z}}{V} \propto \dfrac{\dot{x}}{y^m}$,

$f \cdot \dfrac{\dot{z}}{y^m}$ is a minimum; or $f \cdot \dfrac{\dot{z}}{y^m} \pm f \cdot \dfrac{y\dot{x}}{a^{m+1}}$ is a minimum;

$\therefore \dfrac{y\dot{z}}{a^{m+1}} = \dfrac{\dot{x}}{y^m}$, or $y^{m+1} \dot{z} = a^{m+1} \dot{x}$, the equation to the curve.

COR. 1. Let $m = 0$; then $y\dot{z} = a\dot{x}$, which is an equation to the circle. Here the velocity is uniform.

COR. 2. Let $m = \dfrac{1}{2}$, or the body fall by gravity;

then, $y^{\frac{3}{4}} \dot{z} = a^{\frac{3}{4}} \dot{x}$; $\therefore y^3 \times (\dot{x}^2 + \dot{y}^2) = a^3 \dot{x}^2$; and by re-

duction, $\dot{x} = \dfrac{y^{\frac{3}{2}} \dot{y}}{(a^3 - y^3)^{\frac{1}{2}}}$, the equation to the curve.

Ex. 9. A body moving uniformly from one given point A to another B, is impelled by a force tending to

C, which $\propto \dfrac{1}{CD^2}$, where CD is any variable intermediate distance; to determine the curve, so that its whole action upon the body, or $F \times \dot{t}$, may be a minimum.

Let DCF be a small angle, $Dn = \dot{x}$, $CD = y$, $FD = \dot{z}$.

Then, since the body moves uniformly,

$\dot{t}$, which $\propto \dfrac{\dot{z}}{V} \propto \dot{z}$; $\therefore F \times \dot{t} \propto \dfrac{\dot{z}}{y^2}$ a

minimum. Now the $f. \dfrac{\dot{x}}{y}$ is the measure

of the angle ACB, and is therefore given;

hence, $f. \dfrac{\dot{z}}{y^2} \pm f. \dfrac{\dot{x}}{ay}$ is a minimum; or

$\dfrac{\dot{z}}{ay} = \dfrac{\dot{x}}{y^2}$, and $y \dot{z} = a\dot{x}$, a property of the circle, whose diameter is a.

For if CA be a diameter, CB and CF two chords indefinitely near, and Fd perpendicular to CB, the triangles FBd, BAC, are similar; and $BF : Fd ::$ $CA : CB$, or

$\dot{z} : \dot{x} :: a : y$; $\therefore y \dot{z} = a\dot{x}$.
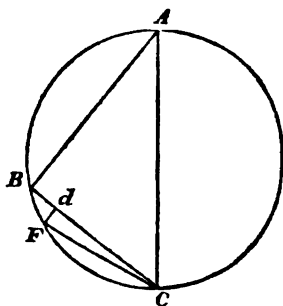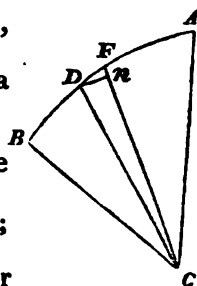
Ex. 10. To determine the curve, in which a body

will move from a given point A to another B in the least time possible ; the velocity being supposed to vary as CD^m.

Here $f.\dot{t}$, or $f.\dfrac{\dot{z}}{v}$, or $f.\dfrac{\dot{z}}{y^m}$ is a minimum ; and the

$f.\dfrac{Dn}{CD}$, or $f.\dfrac{\dot{x}}{y}$, is given ; that is, multiplying the

quantities b and a^m into the denominators of these expressions to make them homogeneous, $f.\dfrac{\dot{z}}{by^m} \pm f.\dfrac{\dot{x}}{a^m y} =$

a minimum ; hence, $by^m \dot{z} = a^m y \dot{x}$, or $by^{m-1}\dot{z} = a^m \dot{x}$, the equation to the curve.

Cor. 1. If ABC be the curve, CB and CF be assumed indefinitely near, and CY be drawn perpendicular on the tangent BY; since $\dfrac{\dot{x}}{\dot{z}} = \dfrac{by^{m-1}}{a^m}$, we have $\dfrac{Fd}{FB}$, or $\dfrac{CY}{CB}$, or $\dfrac{CY}{y} = \dfrac{by^{m-1}}{a^m}$, and $CY = \dfrac{by^m}{a^m}$.
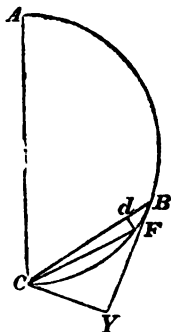
Cor. 2. If $m = 0$, or the velocity be constant, $CY = b$, and the body describes a right line.

Cor. 3. If $m = 1$, $CY = \dfrac{by}{a}$; or $CB : CY :: a : b$ a given ratio, and the curve is the logarithmic spiral.

Cor. 4. If $m = 2$, the curve is a circle ; for here $\dfrac{a^2}{b} \times \dot{x} = y\dot{z}$; its diameter is $\dfrac{a^2}{b}$, and its circumference passes through C.

Ex. 11. Given CA, CB, and the
length of the curve ADB, to find its
nature so that the area may be a max-
imum.
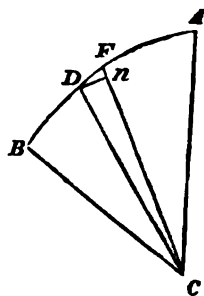
The flux^n. of the area $= \dfrac{CD \times Dn}{2} =$

$\dfrac{y\,\dot{x}}{2}$; and $f.\,\dot{z}$, or $f.\,a\dot{z}$ is given; $\therefore f.\,y\dot{x}$

$\pm f.\,a\,\dot{z}$ is a maximum. Hence $y\dot{z} = a\dot{x}$, and the curve
is a circle, whose diameter is a.

For further information on the subject of this Chapter,
the Reader is referred to the valuable Treatise on *Iso-
perimetrical Problems*, by Mr. Woodhouse.

CHAP. XXIV.

ON THE VARIATION OF TRIANGLES.

(386.) SINCE a triangle consists of six parts, viz. three sides and three angles, any two parts may be supposed constant, while the others admit of a change. If these increments or decrements be small, we may reason about them as the fluxions of the quantities to which they belong. The variations of the sides are represented by the small differences of the sides, and the variations of the angles by the small arcs, which subtend them.

IN RECTILINEAR TRIANGLES.

PROP. 1.   (*Cotes' Theorem* 1.)

(387.) Let ABC be a rectilinear triangle, and let AB and the angle B remain constant; then $\dot{BC} : \dot{AC} ::$ rad. : cos. C.

For, produce BC, and let DAC be a small variation of the angle A, and draw CE perpendicular to AD; then CD and DE are the small increments or fluxions of BC, AC; and $CD : DE :: r :$ cos. ACB.

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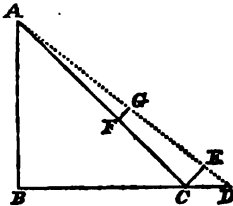
PROP. 2.   (*Cotes' Theor.* 2.)

In the same case, $\dot{A}$, or $\dot{C} : \dot{A}C :: T \angle C : AC.$

For with A as center, and AF a line equal to the tabular radius, describe the small arc FG, the measure of the variation of A.

Then $FG : CE ::$ rad. $: AC$
$$\dot{CE} : DE :: T, D \text{ or } C : r$$
$$\therefore FG : DE :: T \angle C : AC.$$

PROP. 3.   (*Theor.* 3.)

In the same case,

$$\dot{A}, \text{ or } \dot{C} : B\dot{C} :: S \angle C : AC.$$

For $FG : CE :: r : AC$
$$CE : CD :: S, C : r$$
$$\therefore FG : CD :: S, C : AC.$$

PROP. 4.   (*Theor.* 4.)

Suppose BC and the opposite angle A to remain constant.
Then $\dot{C}$, or $\dot{B} : A\dot{B} :: T, C : AB.$

For $\dot{C} : \overline{\text{sine } \angle \dot{C}} :: r : \cos. C$
(Art. 44.)

and $\overline{\text{sine } \angle \tilde{C}} : A\dot{B} :: S, C : AB,$

because $S, C : AB :: S, A : BC :: m : n,$ a given ratio;

$$\therefore \dot{C} : A\dot{B} :: \frac{r \times S, C}{\cos. C} : AB$$

$$:: T, C : AB.$$
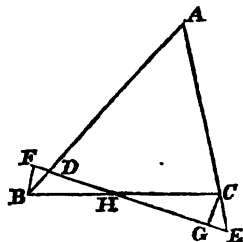
PROP. 5.   (*Theor.* 5.)

In the same case, $\dot{A}B : \dot{A}C ::$ cos. $ACB :$ cos. $ABC.$

For suppose the $\triangle$ ABC to be changed into ADE, then BD is the variation of AB, and CE the variation of AC.

And $BD : DF :: r : \cos. ABC,$

  and $GE : CE :: \cos. ACB : r;$ and $GE = DF,$ since BC is constant;

$$\therefore BD : CE :: \cos. ACB : \cos. ABC.$$

Prop. 6.   (*Theor.* 6.)

Suppose two sides AB and AC to remain constant,

$\dot{A} : \dot{B} :: r \times BC : AC \times \cos. C.$ Let ABC be changed into ABD; draw CE perpendicular to AD; and with B as center, and BH a line equal to the tabular radius, describe the arc HI; then HI is the measure of the variation of the angle B, and FG taken in the same manner is the measure of the variation of A.

Now $FG : CD :: r : AC$
$CD : CE :: r : \cos. C$
$CE : HI :: BC : r;$

$$\therefore FG : HI :: r \times BC : AC \times \cos. C.$$

Prop. 7.   (*Theor.* 7.)

In the same case, $\dot{A} : \dot{BC} ::$ cosec. $C : AC.$

For $FG : CD :: r : AC$
$CD : DE ::$ cosec. $C : r$

$$\therefore FG : DE :: \text{cosec. } C : AC.$$

PROP. 8.   (*Theor.* 8.)

In the same case, $\dot{B} : B\dot{C} :: CoT, C : BC.$

For $HI : CE :: r : BC$
$CE : DE :: CoT, C :: r$
___
$\therefore HI : DE :: CoT, C : BC.$

PROP. 9.   (*Theor.* 9.)

In the same case, $\dot{B} : \dot{C} :: T, B : T, C.$

For the fluxion of an arc or angle $\propto \dfrac{\overline{\text{sine of arc}}}{\overline{\text{cosine}}}$ (Art. 44.); and in this instance the sines of the angles are as the opposite sides, or in a given ratio; therefore the fluxion of the sine $\propto$ the sine; and $\dot{B} \propto \dfrac{\text{sine } B}{\cos. \overline{B}}$, or as $T, B,$ and $\dot{C} \propto T, C.$

PROP. 10.   (*Theor.* 10.)

If two angles be constant, the third is constant; therefore the species of triangle and ratio of the sides is given; and the variations of the sides are proportional to the sides.

IN RIGHT-ANGLED SPHERICAL TRIANGLES.

PROP. 11.

(388.) Let *ABC* be a right-angled spherical triangle, having the angle *C* a right angle; and suppose the angle *B* to remain constant; then

$B\dot{C} : A\dot{C} :: T, A : \cos. AC.$
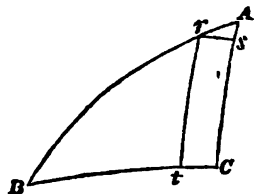
Draw rt perpendicular to BC, and very near AC, and rs perpendicular to AC; $Ar = A\dot{B}$, $As = A\dot{C}$, and $tC = B\dot{C}$; and Ars may be assumed as a small rectilinear triangle.

$$\text{Now} \quad tC : rs :: r : \cos. AC$$
$$rs : sA :: t, A : r;$$
$$\therefore tC : sA :: t, A : \cos. AC.$$

PROP. 12.

In the same case, $\dot{B}A : B\dot{C} :: \cos. AC : S, A.$
$$\text{For} \quad rA : rs :: r : S, A$$
$$rs : tC :: \cos. AC : r$$
$$\therefore rA : tC :: \cos. AC : S, A.$$

COR. Since, by Napier's Rules, $\cos. AC = \dfrac{r \times \cos. AB}{\cos. BC}$,

and $S, A = \dfrac{r \times S, BC}{S, AB}$; $\therefore$

$rA : tc :: \dfrac{\cos. AB}{\cos. BC} : \dfrac{S, BC}{S, AB} :: S, AB \times \cos. AB : S,$

$BC \times \cos. BC :: S, 2 AB : S, 2 BC.$

PROP. 13.

In the same case, $A\dot{C} : A\dot{B} :: \cos. A : r.$
$$\text{For} \quad A\dot{C} : A\dot{B} :: As : Ar.$$

PROP. 14.

Let the hypothenuse AB be supposed constant; then
$$\dot{B} : A\dot{C} :: r : S, BC.$$

Let the triangle ABC be changed into ABE; $DE = A\dot{C}$, and $DC = B\dot{C}$.

The measure of CBE is a small circular arc described with B as the pole, and at the distance of 90°; and this arc, or $\dot{B}$: DE :: r : S, BC.
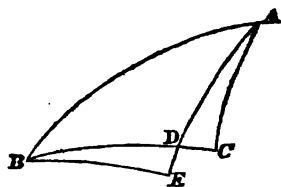
PROP. 15.

In the same case, $B\dot{C} : \dot{A}$ :: S, AC : r; obvious by the last Proposition.

PROP. 16.

In the same case, $B\dot{C} : A\dot{C}$ :: T, AC : T, BC.
For $r \times S, DE = T, BC \times Co T, D$,
and $r \times S, CD = T, AC \times Co T, D$;
$\therefore S, CD : S, DE$ :: T, AC : T, BC;
or, since CD and DE are small,
$CD : DE$ :: T, AC : T, BC.

PROP. 17.

In the same case, $\dot{B}$: $B\dot{C}$ :: $Co T, AC,$: cos. BC.
For $\dot{B}$: DE :: r : S, BC
$DE : CD$ :: $T, BC : T, AC$, by the last Prop.

$$\therefore \dot{B} : CD :: \frac{r \times T, BC}{S, BC} : T, AC$$

$$:: \frac{r^2}{\text{cos. } BC} : T, AC$$

$$:: Co T, AC : \text{cos. } BC.$$

PROP. 18.

In the same case, $\dot{B} : \dot{A} :: \cos. AC : \cos. BC.$

This proportion may be found by compounding the ratios of $\dot{A} : B\dot{C}$, and $B\dot{C} : B$, as already found; or thus, $\quad \dot{B} : DE :: r : S, BC$

$$DE : CD :: T, BC : T, AC$$

$$\underline{CD : \dot{A} :: S, AC : r;}$$

$$\therefore B : \dot{A} :: \frac{S, AC}{T, AC} : \frac{S, BC}{T, BC}$$

$$:: \cos. AC : \cos. BC.$$

PROP. 19.

Let BC be supposed constant, then $\dot{B} : BA :: T, A : S, AB.$

Let the triangle ABC be changed into sBC; and draw sr perpendicular to AB; $Ar = A\dot{B}$; $As = A\dot{C}$; and if B be the pole, the arc intercepted between BA and Bs at the distance of 90° is the measure of the variation ABs.
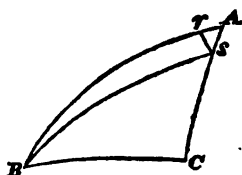
Hence $\quad \dot{B} : rs :: r : S, AB$

$$\underline{rs : rA :: T, A : r;}$$

$$\therefore \dot{B} : rA :: T, A : S, AB.$$

PROP. 20.

In the same case, $\dot{B} : A\dot{C} :: S, A : S, AB.$

For $\quad \dot{B} : rs :: r : S, AB$

$$\underline{rs : As :: S, A : r}$$

$$\therefore \dot{B} : As :: S, A : S, AB.$$

PROP. 21.

In the same case, $A\dot{B} : A\dot{C} :: \cos. A : r$; for $Ar : As$ in that ratio.

IN OBLIQUE SPHERICAL TRIANGLES.

LEMMA.

(389.) If the angular points A, B, C, of the spherical triangle ABC be the poles of three great circles, these great circles will intersect and form another triangle EFG, which is supplemental to the former. Hence, if one of these triangles being changed, the other have a corresponding variation, so that A, B and C shall still be the poles of FG, GE and EF, the variations of the sides in one triangle will be equal to the measures of the variations of the angles opposite to them in the other; and the measures of the variations of the angles will be the corresponding variations of the opposite sides.
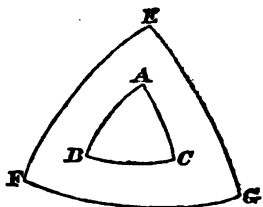
PROP. 22. (*Cotes' Theor.* 11.)

In any spherical triangle ABC, one angle as B and a side adjacent as AB being supposed constant,

$B\dot{C} : A\dot{C} :: r : \cos. C.$

Let ABC be changed into ABD; take $AE = AC$, and join CE. Then CD, DE are the variations of BC, AC;

and $CD : DE :: r : \cos. \angle C.$
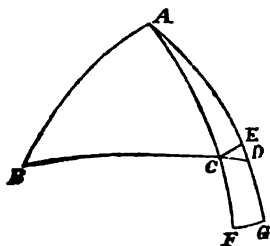
PROP. 23.   (*Theor.* 12.)

In the same case, $\dot{A} : \dot{C} :: r : \cos. AC$.

By the preceding Proposition, in the supplemental triangle *EFG*,

$$FG : E\dot{F} :: r : \cos. F,$$

or $\dot{A} : \dot{C} :: r : \cos. AC.$

PROP. 24.   (*Theor.* 13.)

In the same case,

$$\dot{A} : \dot{AC} :: T, C : S, AC.$$

Take *AF* the arc of a quadrant, and let *FG* be the measure of the variation of *A*;

then $FG : CE :: r : S, AC$

   $CE : DE :: (T, CDE) T, ACB : r$;

$\therefore FG : DE :: T \angle C : S, AC.$

PROP. 25.   (*Theor.* 14.)

In the same case, $\dot{BC} : \dot{C} :: T, AC : S, C.$

In the supplemental triangle, by the last Proposition,

$$\dot{E} : E\dot{F} :: T \angle F : S, EF,$$

or $\dot{BC} : \dot{C} :: T, AC : S, C.$

PROP. 26.   (*Theor.* 15.)

In the same case, $\dot{A} : \dot{BC} :: S, C : S, AC.$

For $FG : CE :: r : S, AC$

   $CE : CD :: S, C, : r$;

$\therefore FG : CD :: S, C : S, AC.$

PROP. 27.   (*Theor.* 16.)

In the same case, $\dot{C} : \dot{AC} :: T, C : T, AC$; for by the three preceding Propositions,

$$\dot{C} : B\dot{C} :: S, C : T, AC$$
$$B\dot{C} : \dot{A} :: S, AC : S, C$$
$$\dot{A} : AC :: T, C : S, AC;$$
$$\overline{\therefore \dot{C} : AC :: T, C : T, AC.}$$

<div align="center">

PROP. 28.   *(Theor.* 17.)

</div>

In any spherical triangle *ABC*, one side as *BC* and the opposite angle *A* being supposed constant,

$$\dot{C} : A\dot{B} :: T, C : T, AB$$
$$\text{For } S, C : S, A :: S, AB : S, BC;$$
$$\therefore S, C \propto S, AB;$$
$$\therefore \dot{S}, C : \dot{S}, AB :: S, C : S, AB.$$

But the fluxion of an arc or angle $\propto \dfrac{\text{the flux. of the sine}}{\text{cosine}}$.

(Art. 44).

$$\therefore \dot{C} : A\dot{B} :: \frac{S, C}{\cos. C} : \frac{S, AB}{\cos. AB} :: T, C : T, AB.$$

<div align="center">

PROP. 29.   *(Theor.* 18.)

</div>

In the same case,

$A\dot{B} : A\dot{C} :: \cos. C : \cos. B.$

Let *ABC* be changed into *ADG*; then since *BC* is constant, $DF = GE$; also $BD = A\dot{B}$, and $CE = A\dot{C}.$
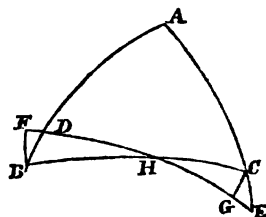
$$\text{Now } BD : DF :: r : \cos. B$$
$$(DF) \; GE : CE :: \cos. C : r;$$
$$\overline{\therefore BD : CE :: \cos. C : \cos. B.}$$

<div align="center">

PROP. 30.   *(Theor.* 19.)

</div>

In the same case, $B : C :: \cos. AC : \cos. AB.$

In the supplemental triangle, by the last case,

$$E\dot{G} : E\dot{F} :: \cos. F : \cos. G,$$

that is, $\dot{B} : \dot{C} :: \cos. AC : \cos. AB.$

PROP. 31. (*Theor.* 20.)

In the same case,

$$A\dot{B} : \dot{B} :: T, AB \times \cos. AB : T, C \times \cos. AC,$$
and as $T, AC \times \cos. C : T, B \times \cos. B.$

For $A\dot{B} : \dot{C} :: T, AB : T, C$ (28.)

$\dot{C} : \dot{B} :: \cos. AB : \cos. AC$ (30.);

$\therefore A\dot{B} : \dot{B} :: T, AB \times \cos. AB : T, C \times \cos. AC.$

Again, $A\dot{B} : A\dot{C} :: \cos. \check{C} : \cos. B$ (29.)

$A\dot{C} : \dot{B} :: T, AC : T, B$ (28.);

$\therefore A\dot{B} : \dot{B} :: T, AC \times \cos. C : T, B \times \cos. B.$

PROP. 32. (*Theor.* 21.)

In any spherical triangle
ABC, the sides *BA* and *AC*
being supposed constant,

$\dot{A} : \dot{B} :: S, BC \times R : S, AC \times \cos. C.$

Let the triangle *ABC* be
changed into *ABD*, *AD*
being equal to *AC*. Draw *CE* perpendicular to *AD*, and
with *A* and *B* as poles at the distance of the arc of a
quadrant let *FG* and *HI* be drawn, the measures of the
variations of *A* and *B*.
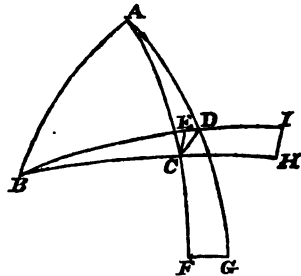
$$FG : CD :: r : S, AC$$
$$CD : CE :: r : \cos. C$$
$$CE : HI :: S, BC : r;$$

$\therefore FG : HI :: r \times S, BC : S, AC \times \cos. C.$

PROP. 33. (*Theor.* 22.)

In the same case, $\dot{A} : B\dot{C}$ :: cosec. $C : S, AC,$
or :: cosec. $B : S, AB.$

For 1st. $FG : CD$ :: r : S, AC
$CD : D\dot{E}$ :: cosec. $C : r;$

$\therefore FG : D\dot{E}$ :: cosec. $C : S, AC.$

2nd, $FG : CD$ :: r : S, AC
$CD : D\dot{E}$ :: r : $S, ACB;$

$\therefore FG : D\dot{E}$ :: $r^2 : S, AC \times S, ACB$
:: $r^2 : S, AB \times S, B$

:: $\dfrac{r^2}{S, B}$: S, AB

:: cosec. $B : S, AB.$

PROP. 34. (*Theor.* 23.)

In the same case, $\dot{B} : B\dot{C}$ :: Co T, C : $S, BC.$
For, $HI : CE$ :: $r : S, BC$
$CE : E\dot{D}$ :: $CoT, C : r;$

$\therefore HI : E\dot{D}$ :: $CoT, C : S, BC.$

PROP. 35. (*Theor.* 24.)

In the same case, $\dot{B} : \dot{C}$ :: T, B : $T, C.$
For since the sines of B and C are as the sines of AC
and AB, the ratio is given;

$\therefore \dot{S}, B : \dot{S}, C$ :: S, B : $S, C.$

But the fluxion of an arc or angle $\propto \dfrac{\text{sine}}{\text{cosine}}$ (Art. 44.);

$\therefore \dot{B} : \dot{C}$ :: $\dfrac{S, B}{\cos. B}$: $\dfrac{S, C}{\cos. C}$ :: T, B : $T, C.$

PROP. 36.   (*Theor. 25.*)

In any spherical triangle, let two of the angles as B and C be supposed constant;

then $B\dot{C} : A\dot{B} :: S, A \times r : S, C \times \cos. AC.$

For, in the supplemental triangle, $EFG,$ by Prop. 32.

$\dot{E} : \dot{G} :: S, FG \times r : S, EF \times \cos. F,$

or $B\dot{C} : A\dot{B} :: S, A \times r : S, C \times \cos. AC.$

PROP. 37.   (*Theor. 26.*)

In the same case, $B\dot{C} : \dot{A} :: \text{cosec.} AB : S, B.$

For, in the supplemental triangle,

$\dot{E} : F\dot{G} :: \text{cosec.} G : S, EG$ (Prop. 33.); $\therefore$ &c.

PROP. 38.   (*Theor. 27.*)

In the same case, $A\dot{B} : \dot{A} :: Co T, AC : S, A.$

For, in the supplemental triangle,

$\dot{G} : F\dot{G} :: C o T, F : S, FG$ (Prop. 34.); $\therefore$ &c.

PROP. 39.   (*Theor. 28.*)

In the same case, $A\dot{B} : A\dot{C} :: T, AB : T, AC.$

For, in the supplemental triangle,

$\dot{G} : \dot{F} :: T, G : T, F$ (Prop. 35.); $\therefore$ &c.

(390.) These propositions are very useful in Astronomy. If in the triangle $ABC,$ A represent the pole of the equinoctial, B the zenith, and C any star, then $\dot{A}$ is the variation of the hour-angle; $\dot{B}$ the variation of the azimuth, and $B\dot{C}$ of the altitude. And the variation of one of these being given, the variation of the other quantities dependent upon it can be found.
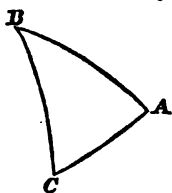
Thus, if a small error in taking the altitude of a known star be given, we can find the corresponding error in time. For here we have $B\dot{C}$ given, to find $\dot{A}$, BA and AC being supposed constant; and by Prop. 33.,

$$\dot{A} : B\dot{C} :: \text{cosec. } B : S, AB;$$

(since cosec. : r :: r : sine) $:: \dfrac{r^2}{S, B} : S, AB;$

$$\therefore \dot{A} = \frac{B\dot{C} \times r^2}{S, B \times S, AB} \propto \frac{1}{S, B};$$

and the error in time is therefore the least, when the star is upon the prime vertical.

CHAP. XXV.

ON CAUSTICS.

SECTION I.

On Caustics by Reflection.

(391.) DEF. IF rays as *BM* be supposed to issue from a luminous point *B*, and to be reflected by the curve *AMD*, so that the angle of incidence equals the angle of reflection, the curve line *HF*, which touches the reflected rays, produced if necessary on the opposite side, is called The Caustic by Reflection.
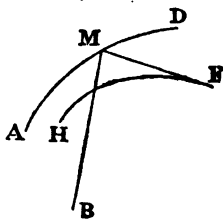
PROP. 1.

To find the length of the caustic *HF*.

Take *BM*, and *Bm*, two rays very near each other, and let them be reflected in the directions *MF*, and *mf* touching the caustic in *F* and *f*, so that *Ff* is a small part of the curve. Draw *Mo* perpendicular to *mf*, and *mr* perpendicular to *MB*. Then as *Mm* is common to the triangles *Mmo*, *mMr*, which are right-angled at *o* and *r*, and have the angle *Mmo=mMr*, the angles of incidence and reflection being equal, we have *mo=Mr*. Hence since the angle at *F* is evanescent, if at *f* a
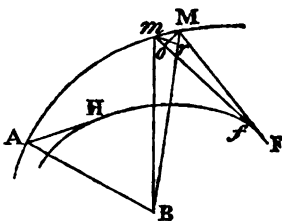
small line ft be conceived to be drawn parallel to Mo, cutting FM in t, $Bm + mf = BM + Mt$;

$$\therefore BM + Mt - Bm - mf = 0.$$

Add Ff to each, and for t write f, Ff and Ft being ultimately equal;

$$\therefore Ff = BM + MF - Bm - mf;$$

and this is true, in whatever part of the curve M is taken; therefore the fluents or sums of these increments are equal; hence if BA be some one ray, and AH the reflected ray touching the curve in H,

$$HF = BM + MF - BA - AH.$$

PROP. 2.

Given the nature of the curve AMD, the luminous point B, and the ray of incidence BM, to find the point F in the reflected ray where it touches the caustic.
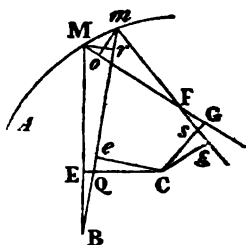
The arc Mm being supposed very small, it is evident that the point F in which the tangents MG, mg intersect each other, is the point in which MG touches the caustic.

Find C the center of the circle of curvature at M; and from C draw CE, Ce, CG, Cg perpendicular to the rays of incidence and reflection, and Mr, mo perpendicular to Bm, GM.

Now in the triangles Mmo, Mmr, $mr = Mo$; and since the sines of incidence and reflection are equal, the triangles CEM, CGM, and Cem, Cgm are respectively equal; $\therefore CE = CG$, and $Ce = Cg$; hence

$CE = Ce = CG - Cg$; that is $EQ = sQ$.

And by similar triangles, BEQ, BMR, and Fmo, FGs,

$$BM + BE \; : \; BM \; :: \; MR + EQ \; : \; Mr$$
$$:: \; mo + Gs \; : \; mo$$
$$:: \; MG \; : \; MF;$$

that is, if BM be assumed $= y$, and ME, or its equal MG be assumed $= a$,

$$2y - a \; : \; y \; :: \; a \; : \; MF; \; \therefore \; MF = \frac{ay}{2y - a}.$$

(392.) If the reflecting curve be convex toward B,

y is negative; and $MF = \dfrac{-ay}{-2y-a} = \dfrac{ay}{2y+a}$.

PROP. 3

To find the caustic, when rays are reflected by a plane surface, or right line.

Let B be the radiant point, and AD a right line in the plane. AD may be considered as a curve, whose radius is infinite; $\therefore$ a is infinite, and $MF = \dfrac{ay}{2y - a} =$

$\dfrac{ay}{-a} = -y = BM$.
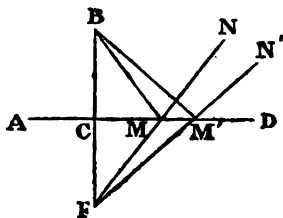
Hence, draw BC perpendicular to AD; produce BC to F, making $CF = CB$; the reflected rays NM, $N'M'$ being produced will meet in F; and the caustic is the point F.

z

Prop. 4.

To find the caustic, when the reflecting curve is the arc of a circle.

Take C the center of the circle, and draw from the luminous point B the line BCD. Let $BM=y$, $EM = a$; then $MF = \dfrac{ay}{2y-a}$. Hence pro-duce MB to O; make $MO = 2y-a$; and MF is a fourth proportional to MO, MB and ME.
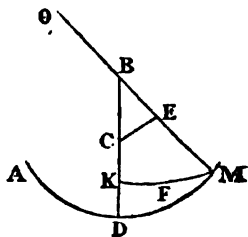
(393.) If y be infinite, $MF=\dfrac{a}{2}$; and the rays, which fall on the arc parallel and near CD, converge to a point K, bisecting CD.

(394.) If $BC = CD$, and the arc be small, $MF = \dfrac{a \times 2a}{3a} = \dfrac{2a}{3} = \dfrac{2CD}{3} = \dfrac{BD}{3}$.

Prop. 5.

Let the reflecting curve ALD be a semi-circle, and the rays of incidence PM be perpendicular to the dia-meter AD; required the caustic AFK.

The radius of the circle CM is the radius of curva-ture, and it is constant. Also, since the rays are parallel, B is at an infinite distance; $\therefore MF$, which equals $\dfrac{ay}{2y-a} = \dfrac{ay}{2y} = \dfrac{a}{2} = \dfrac{MP}{2}$.
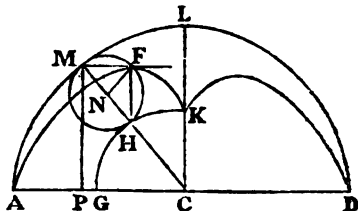
Hence, bisect *CM* in *H*; draw *HF* perpendicular to *MF*, and *F* is a point in the caustic *AFK*; for *MH* : *MC* :: *MF* : *MP*.

(395.) The point *K* bisects *LC*, a radius perpendicular to *AD*.

(396.) The part of the curve $AF = PM + MF = 2MF + MF = 3MF$; and the caustic $AFK = 3CK$.

(397.) When *PMC* is half a right angle, *MF* touches the caustic in the highest point.

(398.) On *MH* as a diameter, describe a circle; it passes through *F*, since *MFH* is a right angle; and if this circle revolve on the quadrant *GHK*, described with radius *CH*, a point *F* in the circumference will trace out the caustic.

For $MH = \frac{1}{4} MC$; and the angle $HNF = 2HMF = 2HCG = 2HCK$; ∴ the arc *HF* = the arc *HK*, and the curve *KFA* is a semi-cycloid, which begins in *K*, and has the extremity of its axis at *A*.

Prop. 6.

To describe the caustic, when the reflecting curve is a semi-circle, and the luminous point in the circumference.

Let *AMD* be the semi-circle, *AD* the diameter, at the extremity of which is the luminous point *A*. Draw *CE* perpendicular to the ray *AM*. Then $AE = EM$;
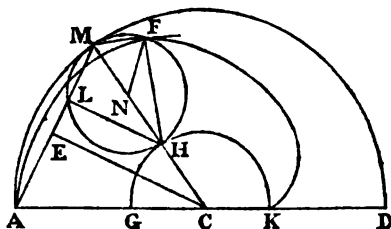
and ∴ *AM*, or $y = 2a$; and *MF*, which $= \dfrac{ay}{2y-a} = \dfrac{1}{3}y$.

z 2

Hence if CH be taken $= \dfrac{CM}{3}$, and HF drawn perpendicular to MF, the point F is in the caustic; for draw HL perpendicular to MA, then $ML = \frac{2}{3} ME = \frac{1}{3} AM$; $\therefore MF = \frac{1}{3} AM$.

(399.) The part of the curve $AF = AM + MF = AM + \frac{1}{3} AM = \frac{4}{3} AM$; and the whole caustic $AFK = \frac{4}{3} AD$.

(400.) If $AM = AC$, MF is parallel to AD, and the point F is the highest point of the caustic.

(401.) On MH as a diameter, describe a circle; it passes through F; and if this circle revolve on the semi-circle GHK described with a radius CG, a point F in the circumference will trace out the caustic.

For in the isosceles $\triangle CMA$, the angle $KCH = 2 CMA = 2 CMF = HNF$; $\therefore$ the arc $HF = HK$, the circles being equal; and the curve KFA is a semi-cycloid, whose vertex is at A.

PROP. 7.

To describe the caustic, when the reflecting curve is a logarithmic spiral, and the rays of incidence issue from the center.

Let AMD be the spiral; A the center, and AM a ray. Draw MC perpendicular to a tangent at M, and AC perpendicular to AM; CM is the radius of curvature;
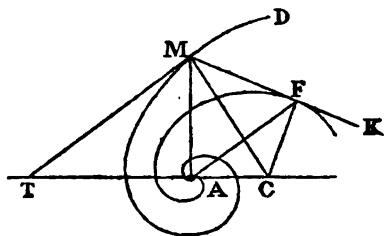
$\therefore AM$, or $y = a$; and $MF \dfrac{a y}{2y - a} = y$, and the triangle

AMF is isosceles. But the $\angle AMT = FMD = MFA$; and AMT is a constant angle in the same spiral; $\therefore$ AFM is constant; and the caustic AF is a logarithmic spiral, differing from the other only in position.

PROP. 8.

Let the reflecting curve be the common semi-cycloid, and let the rays be parallel to the axis; to describe the caustic.

Suppose DMA to be the semi-cycloid, and PM an incident ray; draw MG perpendicular to a tangent at M; then $MG = \frac{1}{2}$ the radius of curvature (Art. 104. Ex. 3.); and
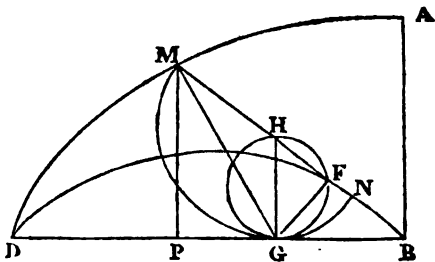
y being infinite, MF, or $\dfrac{ay}{2y-a} = \dfrac{a}{2} = MP$; therefore since the angle $GMP =$ the angle GMF, if GF be drawn perpendicular to MF, the point F will be in the caustic.

(402.) Take H the center of the generating circle; draw HM, HG, to the describing point M, and the point of contact G; HG is perpendicular to DB, and the angle $GMH = MGH = GMP$; hence the reflected ray MF passes through H, the center of the generating circle. Now if a circle be described on GH, it passes through F; $\therefore$ the arcs GN and $\frac{1}{2} GF$, which subtend the same angle GHN at their respective centers, are proportional to the diameters MN and GH, or in the ratio of $2:1$; $\therefore GF = GN = GB$; and the caustic DFB is a cycloid described by the revolution of GFH on the line BD.

(403.) The area $DABFD = 3$ times the area of the circle GFH. For the area of the semi-circle $MGN =$

twice the area of the circle GFH; $\therefore$ the area $AMDB=$ 6 times the area of the circle GFH; and consequently the external area $ADFB=3$ times the area of that circle. Hence the curve DFB divides $AMDB$ into two equal parts.

PROP. 9.

Let the reflecting curve be the common parabola, and the rays come parallel to the axis.

Here y is infinite; $\therefore MF = \dfrac{ay}{2y-a}$ $= \tfrac{1}{2}\, a.$
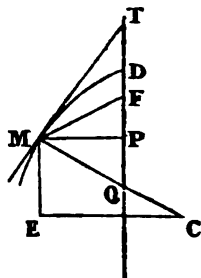
Now in the parabola (Art. 104.) CM the radius of curvature $= \dfrac{(4x + r)^{\frac{3}{2}}}{2\sqrt{r}}$, where x is the abscissa, and r the

latus rectum $=$ (since $rx = BM^2 = v^2$) $\dfrac{(r^2 + 4v^2)^{\frac{3}{2}}}{2\,r^2} =$

$\left(\text{since } QM^2 = PM^2 + PQ^2 = v^2 + \dfrac{r^2}{4}\right) \dfrac{4QM^3}{r^2}.$

But $QM : QP :: CM : ME$ (a)

or $QM : \dfrac{r}{2} :: \dfrac{4QM^3}{r^2} : a;$ $\therefore a = \dfrac{2QM^2}{r}.$

Also, $QT : QM :: QM : (QP)\dfrac{r}{2};$ $\therefore QT = \dfrac{2QM^2}{r}$

$= a.$ But $MF = \dfrac{a}{2};$ $\therefore MF = \dfrac{QT}{2} = QF;$ or the rays converge after reflection to the focus of the parabola.

LEMMA.

(404.) If the luminous point B be in the focus of the parabola, the reflected rays MF are parallel to the

axis; and MF is infinite; $\therefore$.

$\dfrac{ay}{2y-a}$ is infinite, or $2y-a=0$;

$\therefore a = 2y$.

Hence, if MB be produced to E, so that $ME=2MB$, and EC be drawn perpendicular to MC, C is the center of the circle of curvature at M. And if a line be drawn from B, perpendicular to MB, it bisects MC.
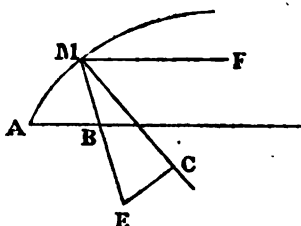
PROP. 10.

Suppose the reflecting curve to be a parabola, and the rays to fall perpendicular to the axis.

If PM be the direction of the incident, and MG of the reflected rays, and from C the center of the circle of curvature, CG be drawn perpendicular to MG, $MF =$

$\dfrac{ay}{2y-a} = \dfrac{a}{2}$.
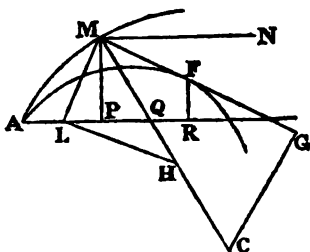
The Figure may be constructed thus:

Draw MN parallel to AP, the axis; and having joined the point M, and the focus L, draw LH parallel to MG; then since $LMC = CMN$, and $PMC=FMC$, $LMF=PMN$; $\therefore$ LMF is a right angle; and the perpendicular LH (by the Lemma) bisects MC; hence

$LH = \dfrac{MG}{2} = \dfrac{a}{2} = MF$. Therefore from M draw MF

equal and parallel to the perpendicular LH, and it will touch the caustic in F.

(405.) When MF is parallel to AP, the ordinate FR is the greatest. In this case the incident ray passes

through the focus, and the angle $PMQ = PQM$, and
$PM = PQ$.   Hence MF, which equals LH, or in this
instance $= PQ$, is equal to half the *latus rectum*; and
AF, the length of the caustic $= PM + MF =$ the *latus
rectum*.

SECTION II.

ON CAUSTICS BY REFRACTION.

(406.) DEF. If rays BM issuing from a luminous point
B be refracted by the curve AMD, so that the sines of
incidence are to the sines of refrac-
tion in a given ratio, the curve
line HFN, which touches all the
refracted rays, is called The Dia-
caustic, or Caustic by Refraction.
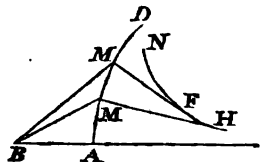
PROP. 1.

To find the Length of the Caustic.

Let BM, Bm
be two rays inci-
dent upon AMD,
very   near   each
other, and suppose
the refracted rays
MG and mg to meet in F. Take C the center of curvature
at M; draw CE, CG perpendicular to BM produced and
MF; and draw Mr, Mo perpendicular to Bm, mg.
Then CE is the sine of incidence of BM, and CG the
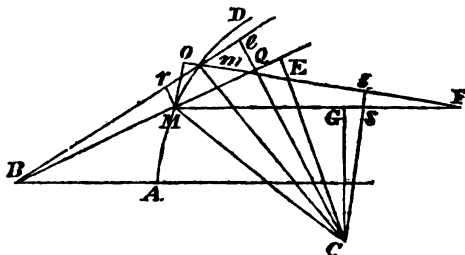
sine of refraction. Let CE be to CG as $m : n$. Now CMm, rME are right angles; take away mME from each, and $rMm = CME$; $\therefore$ the triangles rMm, CME are similar. In the same manner it may be shewn, that mMo, CMG are similar; hence, by compounding the ratios of $CE : CM$, and $CM : CG$, we have

$$CE : CM :: rm : Mm$$
$$\text{and } CM : CG :: Mm : mo:$$
$$\overline{\therefore \ CE : CG :: rm : mo;}$$

or $m : n :: rm : mo$; $\therefore mo = \dfrac{n}{m} \times rm$.

Now suppose mf to touch the curve FN in f, and MF in F; since Mm is indefinitely small, Ff will be a small part of the arc.
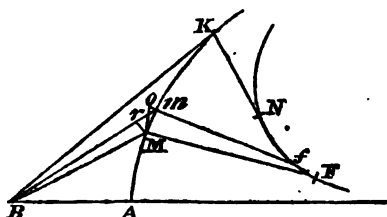
Imagine a small line ft to be drawn perpendicular to MF; then since $FM = Fo$, and $Ff = Ft$, $Mt = fo = mf + mo$; $\therefore Mt - mf - mo = 0$, or $Mt - mf - \dfrac{n}{m}$.

$rm = 0$. Add Ff the fluxion of the curve to each side of the equation; then $Ff = MF - mf - \dfrac{n}{m} . rm$, and the fluents are equal; $\therefore NF = MF - KN - \dfrac{n}{m}(BK - BM)$ the curve being supposed to commence at F.

PROP. 2.

The nature of the curve AMD being given, the luminous point B, and the ray of incidence BM, to find the point F in the refracted ray where it touches the caustic. (See second Fig. in preceding page.)

The point of intersection of two rays MG, mg, which are indefinitely near each other, is the point of contact re-

quired. Draw Ce, Cg perpendicular to Bm produced and mF; the Proposition requires to find MF.

By similar triangles, $MF : MG :: Mo : Mo - gs$, and $MF = \dfrac{MG \times Mo}{Mo - Gs}$.

But by similar triangles,

$$ME : MC :: Mr : Mm,$$
$$\text{and } MC : MG :: Mm : Mo;$$

$$\therefore ME : MG :: Mr : Mo = \frac{MG \times Mr}{ME}.$$

Also, $BM : BQ :: Mr : Qe = \dfrac{BQ \times Mr}{BM}$.

And since

$$Ce : Cg :: CE : CG :: m : n;$$
$$\therefore m : n :: Ce - CE : Cg - CG$$
$$:: Qe : gs = \frac{n \cdot Qe}{m} = \frac{n \cdot BQ \cdot Mr}{m \cdot BM},$$

and $Mo - gs = \dfrac{MG \times Mr}{ME} - \dfrac{n \times BQ \times Mr}{m \times BM}$

$$= \frac{m \times MG \times BM \times Mr - n \times BQ \times ME \times Mr}{m \times BM \times ME};$$

$\therefore MF$, which by the first proportion $= \dfrac{MG \times Mo}{Mo - Gs} =$
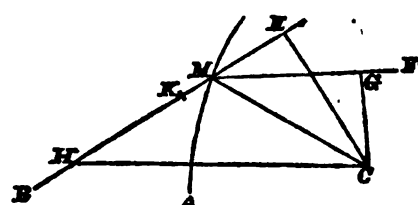
$$\frac{MG \times MG \times Mr}{ME} \div \frac{m \times MG \times BM \times Mr - n \times BQ \times ME \times Mr}{m \times BM \times ME}$$

$$= \frac{m \times MG^2 \times BM}{m \times MG \times BM - n \times BQ \times ME}.$$

Assume $BM = y$, $ME = a$; $MG = b$; then

$$MF = \frac{m\, b^2 y}{m\, b y - n a \cdot (a + y)} = \frac{m\, b^2 y}{m\, b y - n a^2 - n a y}.$$

Hence this construc-
tion.

Towards MC, make
the angle $ECH =$
GCM, and take MK in
the direction $MB =$

$\dfrac{a^2}{y}$; then if MF be a fourth proportional to HK, HE,

MG, the point F will be in the diacaustic.

For, by similar triangles, CGM, CEH,

$$ CG \;:\; CE, \text{ or } n \;:\; m \;::\; MG \;(b) \;:\; EH = \frac{mb}{n}; $$

$$ \therefore HE - ME, \text{ or } HM = \frac{mb}{n} - a = \frac{mb - na}{n}; \text{ and} $$

$$ HM - MK, \text{ or } HK = \frac{mby - nay - na^2}{ny}; \text{ hence,} $$

since $HK : HE :: MG : MF$, we have

by substitution $\dfrac{mby - nay - na^2}{ny} \;:\; \dfrac{mb}{n} \;::\; b \;:\; MF =$

$\dfrac{mb^2 y}{mby - nay - a^2 n}$, or F is a point in the curve

(407.) If the luminous point be on the concave side
of the curve, y is negative; and $MF = \dfrac{-mb^2 y}{-mby + nay - na^2}$

$= \dfrac{mb^2 y}{mby - nay + na^2}$; the construction as before.

(408.) If the luminous point be on the convex side of
the curve, and the value of MF be positive, the points F
and G must be taken on the same side of M. But if the
value of MF be negative, the refracted ray FM must be
produced backward, and the point F must be taken on
the same side of the curve with B.

Hence, in the first case the rays converge after refraction, and in the second case diverge.

(409.) In the same manner if the curve be concave toward B, the refracted rays converge when MF is negative; and diverge when it is positive.

(410.) If the refracting curve be convex toward B, and m less than n, the expression $\dfrac{m\,b^2 y}{mby - nay - na^2}$ is negative, and the refracted rays diverge. If the curve be concave toward B, and m greater than n, that expression is positive; and in this case also the refracted rays diverge.

Prop. 3.

Let the refracting surface be a plane; to determine MF.

Here, since C is at an infinite distance, a and b are infinite; $\therefore$ the expression for MF becomes $\dfrac{m\,b\,y}{\mp na^2}$; and because this quantity is negative, when F and B are on opposite sides of the curve, and positive, when on the same side, F must be taken on the same side with B, and the rays diverge.

The point F may be determined by the following construction;

Draw BO perpendicular to the ray of incidence BM, meeting MC, which is perpendicular to the refracting plane AD in O. Let OL be at right angles to the refracted ray LM; and make $BOH = LOM$. Take MF a fourth proportional to BM, BH, ML; the point F will be in the diacaustic.
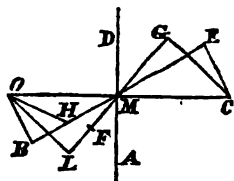
For the triangles *MEC, MBO,* and *MGC, MLO* are similar, whatever be the position of *C*; ∴ when *CM* is infinite,

$$ME \ (a) \ : \ MG \ (b) \ :: \ BM \ (y) \ : \ ML = \frac{b \, y}{a} \ ;$$

and since *OLM, OBH* are similar triangles,

$$OL \ : \ OB \ :: \ \text{or } n \ : \ m \ :: \ ML \ \left(\frac{b \, y}{a}\right) \ : \ BH = \frac{m \, b \, y}{n \, a} \ ;$$

$$\therefore \text{ because } y \ : \ \frac{m \, b \, y}{n \, a} \ :: \ \frac{b \, y}{a} \ : \ \frac{m \, b^2 \, y}{n \, a^2} \ ,$$

$$BM \ : \ BH \ :: \ ML \ : \ MF.$$

Cor. Near the point where the rays fall from *B* perpendicular upon *AD, MF* $= \dfrac{m \, y}{n}$.

Prop. 4.

Let parallel rays fall on the convex side of the quadrant of a circle.

In this case *y* is infinite ; and

$$MF = \frac{m \, b^2}{m b - n a} \ ;$$

and near the extremity of the

diameter, which is parallel to the course of the rays, *b* = *a*; and *MF*, the distance from the vertex *M*, and the point *F*, where the ray cuts the axis $= \dfrac{m \, b}{m - n} = \dfrac{m}{m - n} \times$ radius of the circle.

Hence, if *AMD* be the quadrant, *C* the center, *BM* an incident ray, and *MF* the refracted ray; draw *CD* perpendicular to *BM* produced. Join *CM*; describe

upon it a semi-circle MEC, and take $CE : CG :: m : n$; produce MG to F, making $MF = \dfrac{m\,b^2}{m\,b - a\,n}$; F is the point required.

If $m : n :: 3 : 2$, AH, or the distance between M and F at the vertex $A = 3\,AC$.

If a semi-circle be described on CD, and the chord $CN = \dfrac{2}{3}\,CD$, DN is the course of the extreme ray BD after refraction, and F is in the diacaustic. For in this case $a = 0$; $\therefore MF = b = MG$.

If AP be drawn parallel to CD, the part of the diacaustic FH belonging to the rays refracted between A and $M = AH - MF - \dfrac{2}{3}.(BK - BM)\ PM$, since B is at an infinite distance; and the whole caustic $HFN = 3\,AC - DN - \dfrac{2}{3}\,AC = \dfrac{7}{3}\,AC - DN$.

Cor. If a sphere be generated by the revolution of AMD, all the rays parallel to AC, and near it, converge to a point H, such that $AH = 3\,AC$. And MF for the sphere $= \dfrac{m\,b^2}{m\,b - n\,a}$.

PROP. 5.

Let parallel rays fall on the concave side of the quadrant of a circle.

Here y is infinite; and $MF = \dfrac{m\,b^2}{m\,b - n\,a}$; and near the extremity of

the diameter $a = b$; $\therefore$ MF at the vertex $= \dfrac{m}{m-n} \times$ radius.

Hence, by a construction similar to the last, if $m : n ::$ 2 : 3, and $CG = \dfrac{3\,CE}{2}$, GM produced will be the refracted ray, and $MF = \dfrac{2\,b^2}{2b - 3a}$, a negative quantity.

At the vertex $AH = -2b$; or H falls on the convex side of AMD, and AH is equal to the diameter.

If CG, or $\frac{3}{2}\,CE = CM$, the refracted ray, or the diacaustic touches AMD in M; but $CG : CE :: 3 : 2$; $\therefore$ in this case $CD : CE :: 3 : 2$, or $CE = \frac{2}{3}\,CD$.

The length of a part of the diacaustic $FH = AH -$ $MF + \dfrac{3\,PM}{2}$; and if NK be drawn parallel to DC, N being the point where the refracted ray touches AND, the diacaustic $HFN = 2AC + \frac{3}{2}\,AK = \dfrac{7 + (5)^{\frac{1}{2}}}{2} . AC.$

Cor. If a sphere be generated by the revolution of AMD, all the rays parallel to AC and near it converge to a point H, the distance of which from $A = 2AC$; and MF for the sphere $= \dfrac{m\,b^2}{mb - na}$.

Prop. 6.

Suppose the rays proceeding from a given point B, to fall on the convex side of the quadrant AMD.

Join B and the center of the circle C; and produce BC indefinitely; on CM describe the semi-circle MEC;

produce BM to E, and join CE; take CG $= \dfrac{n}{m} \times CE$; then MG is the refracted ray, and $MF = \dfrac{m\,b^2 y}{mby - na^2 - nay}$.
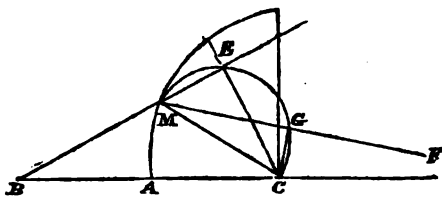
Also, AH taken as before $= \dfrac{m\,b\,y}{my - nb - ny}$.

If r be the radius of a sphere, $MF = \dfrac{mry}{my - nr - ny}$.

<div align="center">

PROP. 7.

</div>

Suppose the rays from a given point B, to fall on the concave side of the quadrant AMD.

In this case y is negative, and the expression for MF is $\dfrac{m\,b^2 y}{mby - nay + na^2}$; and $AH = \dfrac{m\,b\,y}{my - ny + nb}$.
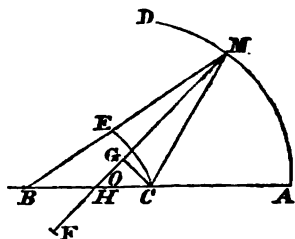
If m be to n as 3 to 2, $AH = \dfrac{3by}{y + 2b}$, and the point F is on the same side of the curve with B; and may be found by this proportion, $AB + 2AC : AC :: 3\,AB :$ distance of F from A.

If rays fall on the concave surface of a sphere, $MF = \dfrac{3ry}{y + 2r}$.

PROP. 8.

To find in the two last cases, where the refracted ray MF meets the axis of the curve.

CASE 1.

Produce MF till it meets BA produced in O. Take $BA = d$; $AO = f$; $BM = y$; $MO = s$; $AR = x$, and $RM = z$.

Then $y = \sqrt{\left(z^2 + (d+x)^2\right)}$
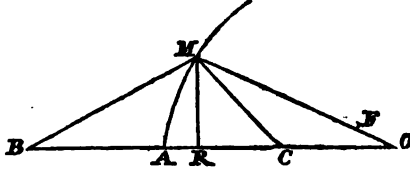
and $s = \sqrt{\left(z^2 + (f-x)^2\right)}$; but $\dot{y} : -\dot{s} :: rm : om$

(Fig. to Prop. 1, page 360.) $:: m : n$; $\therefore n\dot{y} = -m\dot{s}$;

that is, $\dfrac{nz\dot{z} + nd\dot{x} + nx\dot{x}}{\sqrt{\left(z^2 + (d+x)^2\right)}} = \dfrac{mf\dot{x} - mx\dot{x} - mz\dot{z}}{\sqrt{\left(z^2 + (f-x)^2\right)}}$;

from which equation if we expunge $\dot{x}$ or $\dot{z}$, by the nature of the curve f can be found.

In Prop. 6. since $z^2 = 2rx - x^2$, where r is the radius, $z\dot{z} = r\dot{x} - x\dot{x}$; $\therefore$ by substituting for z^2 and $z\dot{z}$ in the preceding equation,

we have $\dfrac{nr + nd}{\sqrt{(2rx + 2dx + d^2)}} = \dfrac{mf - mr}{\sqrt{(2rx - 2fx + f^2)}}$,

from which f can be found.

At the vertex $x = 0$, and $\dfrac{nr + nd}{d} = \dfrac{mf - mr}{f}$; hence

$f = \dfrac{mdr}{md - nd - nr}$, which coincides with the expression, Prop. 6.

CASE 2.

Here B and F are on the same side of the curve.

And, assuming as before,

A A

$$y = \sqrt{\left(z^2 + (d - x)^2\right)}$$
$$s = \sqrt{\left(z^2 + (f - x)^2\right)}$$
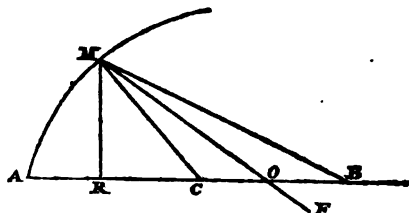
Hence proceeding as in the former case, and substituting $r\dot{x} - x\dot{x}$ for $z\,\dot{z}$, and $2rx - x^2$ for z^2, we have

$$\frac{nr - nd}{\sqrt{2rx - 2dx + d^2}} = \frac{mr - mf}{\sqrt{2rx - 2fx + f^2}},$$ from which f may be found.

At the vertex $x = 0$, and $\dfrac{nr - nd}{d} = \dfrac{mr - mf}{f}$; $\therefore$

$f = \dfrac{mdr}{nr - nd + md}$, which coincides with the expression, Prop. 7.

Chap. XXVI.

SECTION I.

PROBLEMS.

Prob. 1.

Two weights are connected by a cord going over a single fixed pulley ; to determine their ratio, so that one shall generate the greatest quantity of motion possible in the other in a given time.

Let P be the greater weight, and y the less ; then the accelerating force $= \dfrac{P-y}{P+y}$; $\therefore$ the momentum of $y \propto$

$\left(\dfrac{P-y}{P+y}\right) \times y$, which is a maximum by the Problem ;

hence the fluxion of $\dfrac{Py-y^2}{P+y} = 0$, or $(P\dot{y} - 2y\dot{y}) \times (P+y)$

$-\dot{y} \times (Py-y^2) = 0$, or $(P-2y) \times (P+y) = Py - y^2$;
$\therefore y^2 + 2Py = P^2$; and by solving the quadratic,
$y = P \times \sqrt{2-1}.$

$$P \ : \ y \ :: \ 1 \ : \ \sqrt{2} - 1.$$

Prob. 2.

Materials are to be raised to a given height by means of a given weight hanging over a single fixed pulley. What weight must be raised each time, so that the

greatest possible quantity may be raised in a given time?

Let t represent the whole time, n the time of one ascent, S the given space; then $S = mn^2 \times F$; $\therefore n = \left(\frac{S}{mF}\right)^{\frac{1}{2}}$; but $\frac{t}{n} =$ the number of ascents; $\therefore$ this number $= t \times \left(\frac{mF}{S}\right)^{\frac{1}{2}}$. Also, if P be the given power, and x the weight raised each time, $F = \frac{P-x}{P+x}$; $\therefore$ the number $= t \times \left(\frac{m}{S}\right)^{\frac{1}{2}} \cdot \left(\frac{P-x}{P+x}\right)^{\frac{1}{2}}$; and the whole raised $= t \times \left(\frac{m}{S}\right)^{\frac{1}{2}} \times \left(\frac{P-x}{P+x}\right)^{\frac{1}{2}} \times x \propto \left(\frac{P-x}{P+x}\right)^{\frac{1}{2}} \times x$, which by the Problem is a maximum. Hence its square $\frac{Px^2 - x^3}{P+x}$ is a maximum, and its fluxion $= 0$; that is, $(2Px\dot{x} - 3x^2\dot{x}) \times (P+x) - \dot{x} \times (Px^2 - x^3) = 0$, from which equation $x = P \times \frac{\sqrt{5}-1}{2}$.

PROB. 3.

A body falls from A to B by the force of gravity, and then rolls uniformly on a given horizontal line BC with the velocity acquired; to find AB, so that the time of falling down AB added to the time of describing BC may be a minimum.

Let $AB = x$, $BC = b$, $m = 16\frac{1}{12}$ feet. The time down $AB = \left(\frac{x}{m}\right)^{\frac{1}{2}}$, and the velocity acquired $= (4mx)^{\frac{1}{2}}$. Also, since the velocity in BC is uniform, the time of describing $BC = \frac{BC}{v} = \frac{b}{(4mx)^{\frac{1}{2}}}$; $\therefore$

by the Problem, $\left(\frac{x}{m}\right)^{\frac{1}{2}} + \frac{b}{(4mx)^{\frac{1}{2}}}$ is a minimum, or $x^{\frac{1}{2}} +$

$\frac{bx^{-\frac{1}{2}}}{2}$ is a minimum; hence $\frac{\dot{x}}{2x^{\frac{1}{2}}} - \frac{b\dot{x}}{4x^{\frac{3}{2}}} = 0$, and $x = \frac{b}{2}$, or

$AB = \frac{BC}{2}$.

PROB. 4.

From what height above C must a perfectly elastic ball be suffered to descend by gravity, so that it may impinge upon A, and return to the given point C, in the least time possible ?

Let B be the point required; take $BC = x$, $AC = a$; then the time through $BA = \left(\frac{a+x}{m}\right)^{\frac{1}{2}}$; time through

$BC = \left(\frac{x}{m}\right)^{\frac{1}{2}}$; ∴ the time down CA, or the time of

rising through AC after impact, $= \frac{(a+x)^{\frac{1}{2}} - x^{\frac{1}{2}}}{m^{\frac{1}{2}}}$.

Hence the whole time of falling through BA and

returning to $C = \frac{2(a+x)^{\frac{1}{2}} - x^{\frac{1}{2}}}{m^{\frac{1}{2}}}$, a minimum. Hence

$\frac{\dot{x}}{(a+x)^{\frac{1}{2}}} - \frac{\dot{x}}{2x^{\frac{1}{2}}} = 0$, and x is found $= \frac{a}{3}$.

PROB. 5.

Given two perfectly elastic bodies A and B; to find an intermediate elastic body x of such magnitude, that the motion communicated from A to B through x may be a maximum.

Let $a =$ the velocity communicated to A, w the velocity

communicated to x, and b that communicated to B; then, by mechanics,

$$A+x \; : \; 2A \; :: \; a \; : \; w$$
$$x+B \; : \; 2x \; :: \; w \; : \; b;$$
$$\therefore (A+x) \cdot (x+B) \; : \; 4Ax \; :: \; a \; : \; b, \text{ or}$$
$$A+x+\frac{AB}{x}+ B \; : \; 4A \; :: \; a \; : \; b.$$

Now the two mean terms are constant; therefore the first varies inversely as the fourth; but the fourth is a maximum; therefore the first is a minimum. Hence, by taking the fluxion $\dot{x} - \dfrac{AB\dot{x}}{x^2}=0$; $\therefore x^2 = AB$, and A : $x :: x : B$, or x is a mean proportional between A and B.

Cor. If A impinge upon x at rest, x upon y, y upon z, &c. and z upon B, the motion communicated to B is a maximum, when the several bodies interposed are geometric means between the first and the last.

PROB. 6.

Given two sides of a triangle; required the third, so that a body may fall down it by the force of gravity in the least time possible.

Let AB, BC be the two given sides, of which the greater AB is drawn parallel to the horizon; with B as a center, and the less side BC as a radius, describe the circular arc CM. From A draw AF perpendicular to AB, and make $AF = BC$; join FB. Make the angle FBC equal to the angle BFA; produce BC and FA till they meet in D, and join AC; CA is the line required.
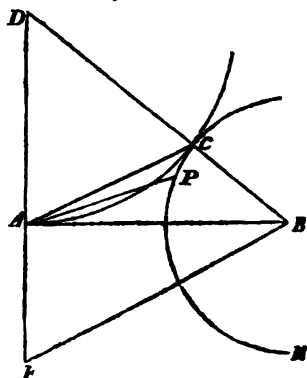
For if BC be produced to D, since $DF = DB$, and $AF = CB$; therefore $DA = DC$; and a circle described, with the center D and radius DA, will touch CM in C. Hence, by Mechanics, the time down CA is less than the time down any other line PA, drawn from A to the circumference CM.

Or fluxionally thus:

Let $AB = a$, $BC = b$, and the perpendicular from C upon $AB = x$; then $AC^2 = a^2 + b^2 - 2a(b^2 - x^2)^{\frac{1}{2}}$; and $\therefore$

$$\frac{a^2 + b^2 - 2a(b^2 - x^2)^{\frac{1}{2}}}{x} = \text{a minimum.}$$ This put into flux-

ions and reduced, gives $x = b \times \dfrac{a^2 - b^2}{a^2 + b^2}$; whence $AC =$

$$\frac{a^2 - b^2}{(a^2 + b^2)^{\frac{1}{2}}}.$$

This result agrees with that which may be deduced from the construction. For since BAF is a right angle, $BF = (BA^2 + AF^2)^{\frac{1}{2}} = (BA^2 + BC^2)^{\frac{1}{2}}$. Also, conceiving a perpendicular to be drawn from A upon BF, the distance of the intersection from $F = \frac{1}{2}(BF - AC)$ will be a third proportional to BF and FA, or to BF and BC.

Hence $\frac{1}{2}(BF - AC) = \dfrac{BC^2}{BF} = \dfrac{BC^2}{(BA^2 + BC^2)^{\frac{1}{2}}}$; $\therefore BF -$

$$AC = \frac{2BC^2}{(BA^2 + BC^2)^{\frac{1}{2}}}; \text{ or } AC = BF - \frac{2BC^2}{(BA^2 + BC^2)^{\frac{1}{2}}}$$

$$= (BA^2 + BC^2)^{\frac{1}{2}} - \frac{2BC^2}{(BA^2 + BC^2)^{\frac{1}{2}}} = \frac{BA^2 - BC^2}{(BA^2 + BC^2)^{\frac{1}{2}}}.$$

PROB. 7.

Given the base of an inclined plane; required its height, so that the time down the plane may be a minimum.

Let the base $= b$, the height $= x$; then the length $=$ $(a^2 + x^2)^{\frac{1}{2}}$; but the time $\propto \dfrac{L}{H^{\frac{1}{2}}} \propto \dfrac{(a^2 + x^2)^{\frac{1}{2}}}{x^{\frac{1}{2}}}$; $\therefore \dfrac{(a^2 + x^2)^{\frac{1}{2}}}{x^{\frac{1}{2}}}$

or $\dfrac{a^2 + x^2}{x}$ is a minimum. Hence $2 x^2 \dot{x} - a^2 \dot{x} - x^2 \dot{x} = 0$,

and $x = a$, or the height $=$ the base.

Prob. 8.

Given the length of an inclined plane; to find its height, when the horizontal velocity of a body, after descending down its length, is a maximum.

Let AB be the plane; produce AB to E, and draw ED perpendicular to the base CB produced.
Take $AB = a$, $AC = x$; then $BC =$ $(a^2 - x^2)^{\frac{1}{2}}$. Now the velocity through $AB = (4 m x)^{\frac{1}{2}}$, if $m = 16\frac{1}{12}$; and this velocity : the horizontal velocity ::
$BE : BD :: BA : BC$
$$:: \quad a \quad : (a^2 - x^2)^{\frac{1}{2}};$$
$\therefore$ the horizontal velocity $\propto x^{\frac{1}{2}} . (a^2 - x^2)^{\frac{1}{2}}$, a maximum.
Hence $a^2 x - x^3$ is a maximum, and $a^2 \dot{x} - 3 x^2 \dot{x} = 0$, and

$x = \dfrac{a}{\sqrt{3}}$, or $AB : AC :: \sqrt{3} : 1$.
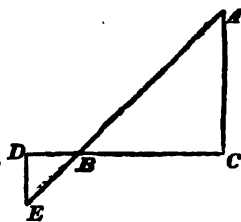
Prob. 9.

Given the height of an inclined plane; required its length, so that a given power acting upon a given weight, in a direction parallel to the plane, may draw it up in the least time possible.

Let $a=$ the height, $x=$ the length, P the power, and W the weight. The tendency of W down the plane is equal to

$\dfrac{W \times a}{x}$, and the accelerating force $= \dfrac{P - \dfrac{W \times a}{x}}{P + W} =$

$\dfrac{P x - W a}{(P + W) \times x}$; but $T^2 = \dfrac{S}{m F}$; $\therefore T^2$ in the case $=$

$\dfrac{x \times (P + W) \times x}{m \times (P x - W a)}$; hence $\dfrac{x^2}{P x - W a}$ is a minimum; $\therefore$

$2 x \dot{x} \times (P x - W a) - P x^2 \dot{x} = 0$; and $P x = 2 a W$, or $P : W :: 2a : x ::$ twice the height : length.

PROB. 10.

A body is projected from a given point, with a given velocity; to find the angle of elevation, when the horizontal range is a maximum.

Let $x=$ the versed sine of twice the angle of elevation; $2r=$ the parameter. The amplitude $\propto$ sine of twice the angle of elevation $\propto (2 r x - x^2)^{\frac{1}{2}}$; $\therefore$ by the Prob. $(2 r x - x^2)^{\frac{1}{2}}$, or $2 r x - x^2$, is a maximum; hence $2 r \dot{x} = 2 x \dot{x}$, and $x = r$, or the angle of projection is 45°.

PROB. 11.

The same supposition being made, it is required to determine the angle of elevation, that the area of the parabola described may be a maximum.

The area of the parabola $= \dfrac{2}{3} \times$ base $\times$ altitude $\propto$ base $\times$ altitude $\propto$ sine $\times$ versed sine of twice the angle of elevation $\propto (2 r x - x^2)^{\frac{1}{2}} \times x$; $\therefore (2 r x - x^2)^{\frac{1}{2}} \times x$, or its square $2 r x^3 - x^4$, is a maximum; hence $6 r x^2 \dot{x} - 4 x^3 \dot{x} = 0$, and

$x = \dfrac{3\,r}{2}$ = versed sine of twice the angle of elevation = versed sine of 120^0; therefore the angle of elevation is 60^0.

PROB. 12.

The same things being given, it is required to deter mine the angle of elevation, when the sum of th amplitude and altitude is a maximum.

In this case, since the altitude $= \dfrac{x}{4}$, we have $\dfrac{x}{4} + (2rx - x^2)^{\frac{1}{2}}$

a maximum; $\therefore \dfrac{\dot{x}}{4} + \dfrac{r\dot{x} - x\dot{x}}{(2rx - x^2)^{\frac{1}{2}}} = 0$; hence $\dfrac{(2rx - x^2)^{\frac{1}{2}}}{4} =$

$x - r$, and x is found in a quadratic equation $= r \times \dfrac{\sqrt{17} \pm 1}{\sqrt{17}}$

= the versed sine of twice the angle of elevation.

PROB. 13.

A body is projected with a given velocity from the top of a given tower; required the direction of projection above an horizontal line AE, that it may fall at the greatest distance possible from the bottom.

Let AB the tower $= a$; suppose the body to be projected in the direction AF, and to fall at D. Let DC parallel to AF meet AB produced in C, and let AE parallel to BD meet the perpendicular DF in E. Take FE or $BC = x$, and p the parameter of the parabola. Then $p \times AC = CD^2$, or $p \times (a+x) = CD^2$; and $DB^2 = DC^2 - CB^2 = p \times (a+x) - x^2$; therefore $p \times (a + x) - x^2$ is a max-imum; and $p\dot{x} = 2x\dot{x}$, or $x = \dfrac{p}{2}$. Hence, in the right-angled triangle AFE, which is similar and equal to DBC, AF and FE are known, from which the angle of elevation FAE may be found.
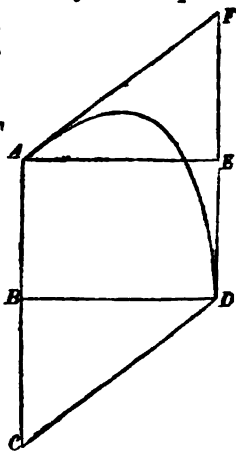
Or geometrically thus;

Let P be the top of the tower; AH the horizontal plane passing through the bottom A of the tower; DP the space due to the velocity with which the ball is projected from P; then is DR, drawn parallel to AH, the common directrix of all the parabolas, that will pass through P. If F be the focus of the parabolic trajectories, then it is known that the distances of the focus from any points in the curve, are equal to the respective vertical distances from those points to the directrix, that is, $FP = PD$, and $FH = HR$. Hence it is evident, that the two points P and H, the latter on AH, will be most remote when F falls in the right line between them; that is, PH is then a maximum; whence, since PA one side of the right-angled triangle PAH is given, the other side AH is a maximum.
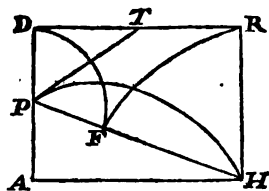
Hence this construction. From P as a center, with the distance $PH = DP + (DP + PA)$, describe an arc to cut AH in H. Draw PH, and then PT to bisect the angle DPH, so will PT be the direction of projection. The horizontal distance $AH = \sqrt{(PH^2 - PA^2)} = \sqrt{((2DP + PA)^2 - PA^2)} = \sqrt{(4DP^2 + 4DP \cdot PA + PA^2 - PA^2)} = 2\sqrt{DP \cdot DA}$. Also, $\dfrac{PH}{PA} =$ sect. HPA;

and $\dfrac{180^\circ - HPA}{2} = DPT$.

Prob. 14.

To deduce a general expression for the time of emptying a vessel through an orifice in the bottom.

Let *AEB* represent the vessel, *C* the orifice, *CD* the perpendicular altitude, and *EHF* the descending surface of the fluid. Take z to represent this surface, n the orifice, $m = 32\frac{1}{6}$ feet, and $x = CG$, the height of the fluid at any point of its descent. Then the velocity of efflux is equal to that which a heavy body would acquire in falling through $\frac{x}{2}$; $\therefore$ the velocity of efflux $= (mx)^{\frac{1}{2}}$. But by hydrostatical principles, as the area of the surface : area of the orifice :: the velocity at the orifice : the velocity of the descending surface; that is,
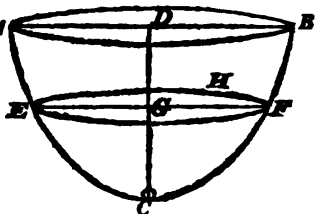
$$z : n :: (mx)^{\frac{1}{2}} : \text{velocity at the surface} = \frac{n \times (mx)^{\frac{1}{2}}}{z};$$

$\therefore \dot{T} = \dfrac{-z\dot{x}}{n\,(mx)^{\frac{1}{2}}}$; the fluent of which, when corrected, gives the time required.

PROB. 15.

1. To find the time in which a given cylinder will empty itself by an orifice in the base.

Here z the descending surface is constant; and if $CD = b$, and $p = 3.14159$, $z = pb^2$; let $BA = a$; then

$$\dot{T} = \frac{-pb^2\dot{x}}{n(mx)^{\frac{1}{2}}} = \frac{-pb^2 x^{-\frac{1}{2}}\dot{x}}{n\,m^{\frac{1}{2}}}; \therefore T = \frac{pb^2}{n\,m^{\frac{1}{2}}} \times$$

$$-2x^{\frac{1}{2}} + \text{corr.} = \frac{2pb^2}{nm^{\frac{1}{2}}}\left(a^{\frac{1}{2}} - x^{\frac{1}{2}}\right); \text{ and}$$

the whole time $= \dfrac{2pb^2 a^{\frac{1}{2}}}{n\,m^{\frac{1}{2}}}$.
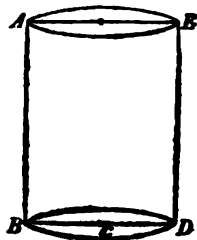
In the same manner the time of emptying any prismatic vessel may be determined.

PROB. 16.

2. To find the time of emptying a hemisphere by a hole in the vertex.

Let ACB represent a hemisphere (See first Fig. in preceding page). Let $DA = a$, $CG = x$; then the descending surface $= p \times GF^2 = p \cdot (2ax - x^2)$; and

$$\dot{T} = \frac{-p}{nm^{\frac{1}{2}}} \times \frac{2ax\dot{x} - x^2\dot{x}}{x^{\frac{1}{2}}} = \frac{-p}{nm^{\frac{1}{2}}} \cdot (2ax^{\frac{1}{2}}\dot{x} - x^{\frac{3}{2}})\dot{x}; \quad \therefore T =$$

$$\frac{-p}{nm^{\frac{1}{2}}} \times \left(\frac{4ax^{\frac{3}{2}}}{3} - \frac{2x^{\frac{5}{2}}}{5} \right) + \text{corr.} \quad \text{Let } x = a, \ T = 0;$$

$$\therefore T = \frac{p}{nm^{\frac{1}{2}}} \times \left(\frac{4a^{\frac{5}{2}}}{3} - \frac{2a^{\frac{5}{2}}}{5} \right) - \frac{p}{n\, m^{\frac{1}{2}}} \times \left(\frac{4ax^{\frac{3}{2}}}{3} - \frac{2x^{\frac{5}{2}}}{5} \right).$$

Let $x = 0$; then the whole time $= \dfrac{14 p \times a^{\frac{5}{2}}}{15 n m^{\frac{1}{2}}}$.

PROB. 17.

3. To find the same, the orifice being made in the base of the hemisphere.

Let $CF = x$; then $FE^2 = a^2 - x^2$; $\therefore p \times FE^2$, or $z = $

$$p \cdot (a^2 - x^2); \quad \therefore \dot{T} = \frac{-p}{nm^{\frac{1}{2}}} \cdot$$

$$\frac{a^2\dot{x} - x^2\dot{x}}{x^{\frac{1}{2}}} = \frac{-p}{nm^{\frac{1}{2}}} \cdot (a^2 x^{-\frac{1}{2}}\dot{x} -$$
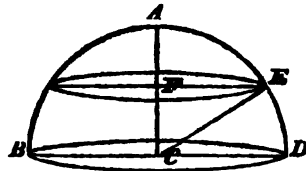
$$x^{\frac{3}{2}}\dot{x}); \quad \text{and } T = \frac{-p}{nm^{\frac{1}{2}}} \cdot \left(2a^2 x^{\frac{1}{2}} - \frac{2x^{\frac{5}{2}}}{5} \right) + C; \quad \text{that is,}$$

$$T = \frac{p}{n\, m^{\frac{1}{2}}} \times : 2a^{\frac{5}{2}} - \frac{2a^{\frac{5}{2}}}{5} - \left(2a^2 x^{\frac{1}{2}} - \frac{2x^{\frac{5}{2}}}{5} \right); \quad \therefore \text{ the}$$

whole time $= \dfrac{8 p a^{\frac{5}{2}}}{5 n m^{\frac{1}{2}}}$.

Cor. If equal hemispheres are emptied by orifices in the vertex and the base, the time in the first case : the time in the last $:: \dfrac{14}{15} : \dfrac{8}{5}$

$$:: 7 : 12.$$

PROB. 18.

4. A sphere ABF being filled with a fluid, it is required to compare the times of emptying the upper and lower hemispheres by an orifice at the bottom D.

Let $DI = x$; then, if $KA = a$, K being the center, $z = p \cdot (2ax - x^2)$; therefore $\dot{T} = \dfrac{-p}{n\,m^{\frac{1}{2}}} \cdot$

$(2a\,x^{\frac{1}{2}}\dot{x} - x^{\frac{3}{2}}\dot{x})$, and $T = \dfrac{-p}{n\,m^{\frac{1}{2}}} \cdot$

$\left(\dfrac{4\,a\,x^{\frac{3}{2}}}{3} - \dfrac{2\,x^{\frac{5}{2}}}{5}\right) + \text{corr.}$ Now if $T = 0$,

$x = 2a$; $\therefore T = \dfrac{p}{n m^{\frac{1}{2}}} \cdot \left(\dfrac{4\,a \times (2a)^{\frac{3}{2}}}{3} - \dfrac{2 \times (2a)^{\frac{5}{2}}}{5}\right) -$

$\dfrac{p}{n\,m^{\frac{1}{2}}} \cdot \left(\dfrac{4\,a\,x^{\frac{3}{2}}}{3} - \dfrac{2\,x^{\frac{5}{2}}}{5}\right)$; or the whole time of emptying

$= \dfrac{p\,a^{\frac{5}{2}}}{n\,m^{\frac{1}{2}}} \cdot \dfrac{16 \times 2^{\frac{1}{2}}}{15}$, since x vanishes.

Again, if $x = a$, we get the time of emptying the lower hemisphere $= \dfrac{p\,a^{\frac{5}{2}}}{n\,m^{\frac{1}{2}}} \cdot \dfrac{14}{15}$. Hence the time of emptying

the upper part $= \dfrac{16 \times 2^{\frac{1}{2}} - 14}{15} \times \dfrac{p\,a^{\frac{5}{2}}}{n\,m^{\frac{1}{2}}}$; $\therefore$ the time of

emptying the upper part : time of emptying the lower
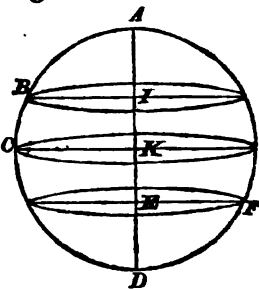
$$:: \frac{16 \times 2^{\frac{1}{2}} - 14}{15} : \frac{14}{15} :: 8 \times 2^{\frac{1}{2}} - 7 : 7.$$

PROB. 19.

5. Let the vessel be a paraboloid, and the orifice in the vertex.

Let $c =$ the *latus rectum;* then, if $DE = x$, $FE^2 = cx$,

and $x = pcx$; $\therefore \dot{T} = \frac{-pc}{n \, m^{\frac{1}{2}}} \times \frac{x \dot{x}}{x^{\frac{1}{2}}} =$

$$\frac{-pc}{n \, m^{\frac{1}{2}}} \times x^{\frac{1}{2}} \dot{x} ; \therefore T = \frac{-pc}{n \, m^{\frac{1}{2}}} \times \frac{2 x^{\frac{3}{2}}}{3}$$

+ corr. Let $x = DC = a$; then $T = 0$; $\therefore$ the whole time

$$= \frac{pc}{n \, m^{\frac{1}{2}}} \times \frac{2 a^{\frac{3}{2}}}{3} .$$
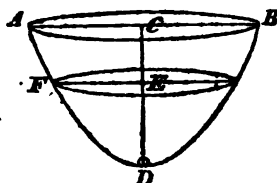
PROB. 20.

6. Let the vessel be a paraboloid, and the orifice in the base.

Let $CE = x$, $DE = a - x$; $\therefore pc.(a - x) = x$; $\therefore \dot{T} =$

$$\frac{-pc}{n \, m^{\frac{1}{2}}} . \left(\frac{a \dot{x} - x \dot{x}}{x^{\frac{1}{2}}} \right) = \frac{-pc}{n \, m^{\frac{1}{2}}} . (ax^{-\frac{1}{2}} \dot{x} - x^{\frac{1}{2}} \dot{x}), \text{ and }$$

$$T = \frac{-pc}{n \, m^{\frac{1}{2}}} . \left(2 a x^{\frac{1}{2}} - \frac{2 x^{\frac{3}{2}}}{3} \right) + \text{corr.}$$

Now if $x = a$, $T = 0$; hence the whole time of emptying $= \frac{pc}{n \, m^{\frac{1}{2}}} \times$

$$\frac{4 a^{\frac{3}{2}}}{3} .$$
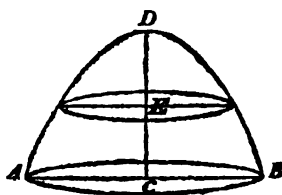
Cor. If the paraboloids in the last Examples are equal, the time of emptying the former : time of emptying the latter :: $\dfrac{2}{3}$: $\dfrac{4}{3}$

:: 1 : 2.

PROB. 21.

7. Let ADB be a cone, and the orifice in the vertex. Let $DC = a$, $CB = b$, $DG = x$, $GF = y$; then $y = \dfrac{bx}{a}$, and z, or $py^2 = \dfrac{pb^2x^2}{a^2}$; $\therefore \dot{T} = \dfrac{-pb^2}{a^2n\,m^{\frac{1}{2}}} \times x^{\frac{3}{2}}\dot{x}$; and the whole time of emptying $= \dfrac{pb^2}{a^2nm^{\frac{1}{2}}} \times$

$\dfrac{2\,a^{\frac{5}{2}}}{5} = \dfrac{2pb^2a^{\frac{1}{2}}}{5\,n\,m^{\frac{1}{2}}}$.
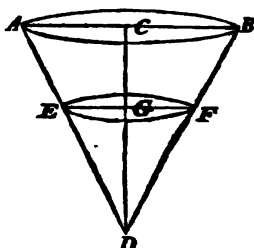
Cor. The time of emptying a cone at the vertex : time of emptying a cylinder of the same base and altitude :: $\dfrac{2}{5}$: 2 :: 1 : 5. (See Prob. 15.)

PROB. 22.

8. To find the time, when the orifice is in the base. Here let $CG = x$; $\therefore DG = a - x$; then z, or $p \times y^2 = \dfrac{pb^2}{a^2}\cdot(a^2 - 2ax + x^2)$;

$\therefore \dot{T} = -\dfrac{pb^2}{a^2\,nm^{\frac{1}{2}}}\cdot(a^2x^{-\frac{1}{2}}\dot{x} - 2a\,x^{\frac{1}{2}}\dot{x} + x^{\frac{3}{2}}\dot{x})$; and the whole time $= \dfrac{16}{15} \times \dfrac{pb^2\,a^{\frac{1}{2}}}{nm^{\frac{1}{2}}}$.
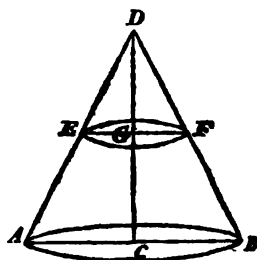
Cor. If the cones be equal in the two last cases, the time of emptying the former : time of emptying the latter :: $\dfrac{2}{5}$: $\dfrac{16}{15}$.

:: 3 : 8.

Let CAD represent a plane figure or a solid, generated by the revolution of CAD about its axis AK. It is required to compare the resistance of the curve line CAD, and of the surface of the solid, with the resistance on the base CD; the plane figure or solid being supposed to move in a fluid in the direction of its axis.

Take FE a small arc; draw FH, EG, perpendicular to CD, and FQ, EP, perpendicular to AK. Let $AQ=x$, $QF=y$, $AF=z$; then ultimately $FE=\dot{z}$, $FR=\dot{x}$, and $ER=\dot{y}$. Let LF represent the force of one particle of water ; draw LM perpendicular to a tangent FM, and MN to LF; then, if this particle struck the base perpendicularly at H, it would be wholly effective ; but its force on the curve at F is diminished in the ratio of LF to LN, or of LF^2 : LM^2; that is, of FE^2 : ER^2, or of $\dot{z}^2$: $\dot{y}^2$.
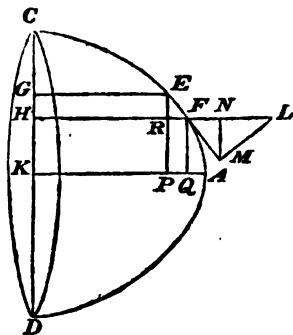
Now, let CAD be considered first as a plane surface. The number of particles which strike upon FE and HG is the same, and it varies as GH or $\dot{y}$. But the whole effect is as the number of particles × the force of each ; ∴ the effect or the resistance on FE : resistance on

$$GH :: \frac{\dot{y}\dot{y}^2}{\dot{z}^2} : \dot{y} :: \frac{\dot{y}^3}{\dot{z}^2} : \dot{y} :: \frac{\dot{y}^3}{\dot{x}^2+\dot{y}^2} : \dot{y}, \text{ or } :: \frac{\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}} : \dot{y};$$

$\therefore$ the whole resistance on the curve CAD : that on the base $CD :: f . \dfrac{\dot{y}}{1 + \dfrac{\dot{x}^2}{\dot{y}^2}} : f . \dot{y}.$

In the next place, suppose the figure to be a solid, generated by the revolution of CAD about its axis. In this case, the number of particles striking on the annulus, formed by the revolution of the part GH, is proportional to that annulus, or to the product of GH and the circumference described by H; that is, it varies as $2py\dot{y}$, or as $y\dot{y}$. Hence, since the resistance is proportional to the number of particles × the force of each, the resistance on the surface : that on the base :: $f . \dfrac{y\dot{y}^3}{\dot{s}^2} : f . y\dot{y}.$

$$:: f . \frac{y\dot{y}^3}{\dot{y}^2 + \dot{x}^2} : f . y\dot{y} :: \frac{f . y\dot{y}}{1 + \dfrac{\dot{x}^2}{\dot{y}^2}} : f . y\dot{y}.$$

PROB. 24.

1. To compare the resistance upon the sides of an isosceles triangle or prismatic solid CAD, moving in the direction of a perpendicular AK, with that on the base.

Here if $AC = s$, $CK = b$, $FQ = y$, $FE = \dot{s}$, $ER = \dot{y}$, $\dot{z}^2 :$ $\dot{y}^2 :: s^2 : b^2$; $\therefore$ since the resistance on the sides : that on the base :: $f . \dfrac{\dot{y}\dot{y}^2}{\dot{s}^2} : f . \dot{y}$ (Prob. 23), it is :: $f . \dfrac{\dot{y}b^2}{s^2} : f . \dot{y} :: b^2 : s^2 :: KC^2 : CA^2.$
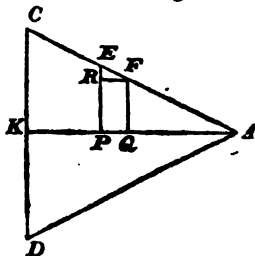
COR. If CAD be a right angle, and therefore $ACK = 45°$, the resistance on the sides : that on the base :: $1 : 2$.

PROB. 25.

2. Let CAD be a semi-circle moving in the direction KA.

Here $\dot{z} = \dfrac{a\dot{y}}{(a^2 - y^2)^{\frac{1}{2}}}$ (Art. 44.); $\therefore \dot{z}^2 = \dfrac{a^2\dot{y}^2}{a^2 - y^2}$. Now the

resistance on CAD : the resistance on $CD :: f . \dfrac{\dot{y}^3}{\dot{z}^2} : f . \dot{y}$;

$\therefore$ in this case, they are $:: f . \dfrac{\dot{y}^3 \cdot (a^2 - y^2)}{a^2 \dot{y}^2} : f . \dot{y} ::$

$f . \dfrac{a^2 \dot{y} - y^2 \dot{y}}{a^2} : f . \dot{y} :: y - \dfrac{y^3}{3\,a^2} : y :: (\text{if } y = a)\ a - \dfrac{a}{3} :$

$a :: 2 : 3.$

PROB. 26.

3. Let CAD represent a cone.

Here if $CK = b$, and $AC = s$, $\dfrac{\dot{y}^2}{\dot{z}^2} =$

$\dfrac{b^2}{s^2}$; $\therefore$ since resistance on surface :

resistance on base $:: f . \dfrac{y \times \dot{y}^3}{\dot{z}^2} : f . y\dot{y}$,

we have, res. on surface : res. on

base $:: f . y\dot{y} \times \dfrac{b^2}{s^2} : f . y\dot{y} :: b^2 : s^2$

$:: CK^2 : CA^2.$
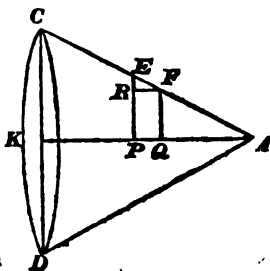
Cor. 1. If CAD be a right angle the former resistance : the latter $:: 1 : 2.$

Cor. 2. If a right-angled cone and cylinder, whose bases are the same, move in the same fluid in the direction of their axes, and the resistance upon the cone = the

resistance on the cylinder; then the velocity of the cone : velocity of the cylinder :: $2^{\frac{1}{3}}$: 1. For let V represent the velocity of the cone, and v of the cylinder; then, since by the last Cor. when the velocities are the same, the resistances are as b^{2} : s^{2}, and that, *cæteris paribus*, the resistance varies as the square of the velocity; ∴ in this case, resistance on the cone : resistance on the cylinder :: $V^{2}\,b^{2}$: $v^{2}\,s^{2}$; whence $V^{2}\,b^{2} = v^{2}\,s^{2}$, and V : v :: s : b :: $2^{\frac{1}{3}}$: 1.

Prob. 27.

4. To compare the resistances upon a globe and cylinder of equal diameters, moving with equal velocities in the direction of the axis of the cylinder.

In general, the fluxion of the resistance on the solid : the fluxion of that on the base :: $\dfrac{y\,\dot{y}^{3}}{\dot{x}^{2}}$: $y\dot{y}$, since $\dot{x}^{2} =$

$\dfrac{a^{2}\dot{y}^{2}}{a^{2}-y^{2}}$, (Art. 44.) :: $\dfrac{(a^{2}-y^{2})\cdot y\dot{y}}{a^{2}}$: $y\dot{y}$,

:: $y\dot{y} - \dfrac{y^{3}\dot{y}}{a^{2}}$: $y\dot{y}$; ∴ the whole re-

sistance on the globe : that on the cylinder :: $\dfrac{y^{2}}{2} - \dfrac{y^{4}}{4\,a^{2}}$: $\dfrac{y^{2}}{2}$; if $y = a$, :: $\dfrac{2\,a^{2}-a^{2}}{4}$: $\dfrac{a^{2}}{2}$ :: 1 : 2.

Prob. 28.

5. Let the solid be the common paraboloid moving in the direction of its axis.

Here $y^{2} = ax$; ∴ $\dot{x} = \dfrac{2\,y\dot{y}}{a}$, and $\dfrac{\dot{x}^{2}}{\dot{y}^{2}} = \dfrac{4\,y^{2}}{a^{2}}$; hence, since

resistance on solid : resistance on the base :: $\displaystyle f.\ \dfrac{y\dot{y}}{1 + \dfrac{\dot{x}^{2}}{\dot{y}^{2}}}$:

$f.y\dot{y}$, they are $:: f. \dfrac{y\dot{y}}{1+\dfrac{4y^2}{a^2}} : f.y\dot{y} :: \dfrac{a^2}{8} \times$ hyp. log.

$\left(\dfrac{a^2}{4}+y^2\right) : \dfrac{y^2}{2}$

(Art. 42. Ex. 5.) $:: \dfrac{a^2}{4}\times$ hyp. log. $\left(\dfrac{a^2}{4}+y^2\right) : y^2.$
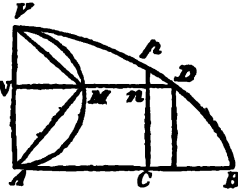
PROB. 29.

6. A solid, generated by the revolution of the common cycloid about its base, moves in a fluid in the direction of its base; to compare the resistance against this solid with that against its circumscribing cylinder.

Let $VA=2a$, $BC=x$, $CD=y$; then $\dot{x} : \dot{y} :: Dn : np$

$:: MN : NV :: AN^{\frac{1}{2}} : NV^{\frac{1}{2}}$, or

$\dot{x}^2 : \dot{y}^2 :: y : 2a-y$; $\therefore \dfrac{\dot{x}^2}{\dot{y}^2}=\dfrac{y}{2a-y}$;

hence resistance on solid : resistance on

the cylinder $:: f. \dfrac{y\dot{y}}{1+\dfrac{y}{2a-y}} : f.y\dot{y} ::$

$f. \dfrac{2ay\dot{y}-y^2\dot{y}}{2a} : f.y\dot{y} :: \dfrac{y^2}{2} - \dfrac{y^3}{6a} : \dfrac{y^2}{2}$; that is, if $y=2a$,
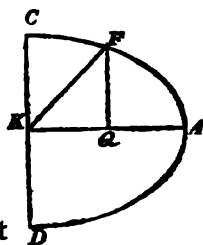
resistance on the solid : resistance on the cylinder : $2a^2 - \dfrac{4a^2}{3} : 2a^2 :: 1 : 3$.

PROB. 30.

7. Let CAD be a spheroid, whose center is K, moving in the direction of the major axis KA; to compare the resistances as before.

Let $AQ=x$, $KQ=u$, $QF=y$; then, if $AK=a$, and the *latus rectum* $=2r$, by a property of conic sections, $a^2 - u^2 : y^2 ::$

$a : r$; $\therefore a^2 - u^2 = \dfrac{a y^2}{r}$, and $u=$

$\left(a^2 - \dfrac{a y^2}{r}\right)^{\frac{1}{2}}$, and $\dot{u} = \dfrac{-a y \dot{y}}{r \cdot \left(a^2 - \dfrac{a y^2}{r}\right)^{\frac{1}{2}}}$; but

since $u+x=a$, $\dot{u}=-\dot{x}$; $\therefore \dot{x} = \dfrac{a y \dot{y}}{r \cdot \left(a^2 - \dfrac{a y^2}{r}\right)^{\frac{1}{2}}}$, and $\dfrac{\dot{x}^2}{\dot{y}^2}=$

$\dfrac{a^2 y^2}{a^2 r^2 - a r y^2} = \dfrac{a y^2}{a r^3 - r y^3}$, and $1 + \dfrac{\dot{x}^2}{\dot{y}^2} = 1 + \dfrac{a y^3}{a r^3 - r y^2} =$

$\dfrac{a r^2 + (a-r) \cdot y^2}{a r^2 - r y^2}$; therefore $\dfrac{y \dot{y}}{1 + \dfrac{\dot{x}^2}{\dot{y}^2}} = \dfrac{a r^2 - r y^2}{a r^2 + (a-r) \cdot y^2} \times y \dot{y}$,

and the fluent (by Fluent 10.) $= \dfrac{-r y^2}{2 \cdot (a-r)} + \dfrac{\frac{1}{2} a^2 r^3}{(a-r)^2} \times$ hyp.

log. $\left(y^2 + \dfrac{a r^2}{a-r}\right) = A$; $\therefore$ resistance against the

spheroid : resistance on the cylinder $:: A : \dfrac{y^2}{2}$

$:: 2A : y^2$.

Prob. 31.

8.   If the figure be an hyperboloid, by conic sections,

$u^2 - a^2 : y^2 :: a : r$; and in this case, $\dfrac{y \dot{y}}{1 + \dfrac{\dot{x}^2}{\dot{y}^2}} =$

$\dfrac{(r^2 a + r y^2) \cdot y \dot{y}}{r^2 a + (a+r) y^2}$, whose fluent $= \dfrac{\frac{1}{2} r y^2}{r+a} + \dfrac{\frac{1}{2} r^2 a^2}{(r+a)^2} \times$ hyp.

log. $\left(y^{\cdot} + \dfrac{r^{2}a}{r+a}\right) = B$ (Fluent 10.); $\therefore$ resistance against

the solid : resistance on the cylinder :: $2B : y^{\cdot}$.

PROB. 32.

9. To determine the frustum $CDEF$ of a triangular prism, of a given base CF, and altitude BA, which, moving in a medium in the direction of its length BA, shall be resisted the least possible.

Let CH drawn parallel to AB meet ED produced in

H; on HC describe a semi-circle, meeting DC in P; join HP, and draw PQ perpendicular to HC, and PR to DH. Then the resistance to DC, which varies as the number of particles $\times$ the force of each, varies as $\dfrac{DH \times DH^{\cdot}}{DC^{2}}$, or as $\dfrac{DH \times DR^{2}}{DP^{3}}$; but
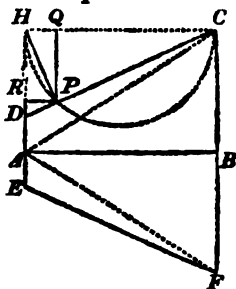
$DH : DP :: DP : DR$, since the angle DPH is a right angle; $\therefore DH : DR :: DH^{2} : DP^{2}$, or :: $DP^{2} : DR^{2}$; $\therefore DR = \dfrac{DH \times DR^{2}}{DP^{2}}$, or the resistance to

DC varies as DR; and the resistance to AD and DC varies as $AD + DR$. Now this is a minimum, when DR is a minimum, or when RH is a maximum; but $RH = PQ$, and PQ is a maximum (Art. 23. Ex. 2.) when $CQ = QH$; that is, when DCH, or $DCB = 45°$.

COR. In this case BC is supposed to be greater than BA; if not, the whole prism will be less resisted than any frustum $CDEF$ of a greater prism.

PROB. 33.

10. To determine that frustum of a cone of a given

base and altitude, which, moving in the direction of its axis, shall be less resisted than any other.

Let $HBEI$ be the required frustum, and HAI the complete cone. Then, by Prob. 26, the resistance on the surface AH : that on the base HI :: HG^2 : AH^2; ∴ if the resistance on the base $= 1$, that on the surface $= \dfrac{HG^2}{AH^2}$.
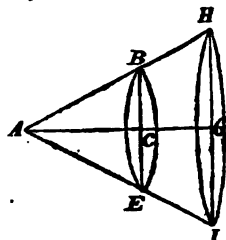
Also, since the resistance varies as the number of particles into the force of each, the resistance on the base HI : that on the base BE :: HG^2 : BC^2, the force of each being the same when the velocity is the same; ∴ the resistance on the base $BE = \dfrac{BC^2}{HG^2} = \dfrac{AB^2}{AH^2}$; and the resistance on the base BE : that on the surface BAE :: AB^2 : BC^2; therefore the resistance on the surface $BAE = \dfrac{BC^2}{HG^2} \times \dfrac{HG^2}{AH^2} = \dfrac{BC^2}{AH^2}$. Hence the resistance on the surface of the frustum $BHIE = \dfrac{HG^2 - BC^2}{AH^2}$;

and the whole resistance $= \dfrac{AB^2 + HG^2 - BC^2}{AH^2} = \dfrac{AC^2 + HG^2}{AH^2}$.

Let $AG = x$, $CG = a$, $HG = b$; then $AH^2 = b^2 + x^2$;

and the resistance $= \dfrac{(x - a)^2 + b^2}{x^2 + b^2} = \dfrac{x^2 - 2ax + a^2 + b^2}{x^2 + b^2}$

$= 1 + \dfrac{a^2 - 2ax}{x^2 + b^2}$, which is a minimum;

∴ $\dfrac{-2a\dot{x}.(x^2 + b^2) + 2x\dot{x}.(2ax - a^2)}{(x^2 + b^2)^2} = 0$;   hence

$x^2 + b^2 = 2x^2 - ax$; $\therefore x^2 - ax = b^2$, and $x = \dfrac{a}{2} \pm$

$\left(\dfrac{a^2}{4} + b^2\right)^{\frac{1}{2}}$.

Cor. The Thirty-second Problem might be solved by the same process; the only difference is, that the resistance on the base CF in that Problem : that on DE :: CB : DA; and the expression, which is a minimum, $=$ $\dfrac{LA}{LB} + \dfrac{BC^2}{LC^2} \times \left(1 - \dfrac{LA}{LB}\right)$, BA and CD being produced to meet in L, or $\dfrac{LA}{LB} + \dfrac{BC^2}{LC^2} \times \dfrac{AB}{LB}$; which according to the same substitution, varies as $\dfrac{x - a}{x} +$ $\dfrac{b^2 a}{x^3 + b^2 x}$; and the fluxion of this quantity being put $= 0$, gives $x = b$, or the angle $DCB = 45°$, as before.

Prob. 34.

To find at what angle the wind must act upon the sails of a mill, so that its effect to produce motion may be a maximum.

Let $x =$ the cosine of the angle; then if $r =$ radius, the sine $= (r^2 - x^2)^{\frac{1}{2}}$; but the effect by hydrostatics $\propto$ $x.(r^2 - x^2)$; or $r^2 x - x^3$ is a maximum; hence $r^2 \dot{x} - 3 x^2 \dot{x}$ $= 0$, and $x = r \times \dfrac{1}{3^{\frac{1}{2}}}$, the cosine of $54°.44'$.

Prob. 35.

Let the triangle ABC be immersed in a fluid, so that its base may be level with the surface; to find where a

line DE must be drawn parallel to the base, so that the pressure upon it may be a maximum.

Draw CF perpendicular to AB; let $AB=a$, $CF=b$, $FG=x$; then $CF : CG :: AB : DE$,

or $b : b-x :: a : DE, =a \times \dfrac{(b-x)}{b}$.
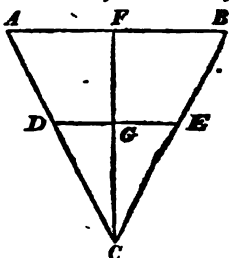
Now the pressure varies as the surface pressed into the depth of the center of gravity; ∴ the pressure in this case varies as $(b-x) \times x \propto bx - x^2$, a maximum; hence $b\dot{x} - 2x\dot{x} = 0$, and $x = \dfrac{b}{2}$.

PROB. 36.

If a semi-circle ACB be immersed vertically in a fluid, with its diameter contiguous to the surface; to find on which of the chords parallel to AB there is the greatest pressure, the density of the fluid being supposed to vary as the depth.

Let $CD=x$, $CE=r$; then the pressure on FE varies as $FE \times$ depth $\times$ density $\propto 2(r^2 - x^2)^{\frac{1}{2}} \times x^2$, a maximum; hence the flux-ion of $r^2 x^4 - x^6 = 0$, or $4r^2 x^3 \dot{x} = 6 x^5 \dot{x}$; ∴ $x^2 = \dfrac{4r^2}{6} - \dfrac{2r^2}{3}$, and $x = r$.
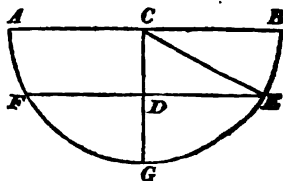
$\left(\dfrac{2}{3}\right)^{\frac{1}{2}}$.

If AGB be a parabola, on the same supposition, $x = \dfrac{4b}{5}$, b being assumed $= CG$.

PROB. 37.

Let AGB represent a hemisphere; to find the section FE parallel to the surface, on which the pressure is a
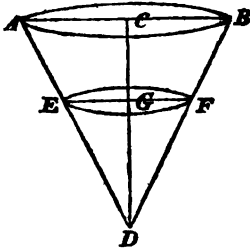
maximum, the density of the fluid being as the nth power of the depth.

The area of the section FE varies as y^2, or as $(r^2 - x^2)$; hence the pressure varies as $(r^2 - x^2) \cdot x \times x^n$; that is, $r^2 x^{n+1} - x^{n+3}$ is a maximum; hence $(n+1) \cdot r^2 x^n \dot{x} = (n+3) \cdot x^{n+2} \dot{x}$, and $x = \left(\dfrac{n+1}{n+3}\right)^{\frac{1}{2}} \cdot r$.

PROB. 38.

A cubic inch of metal, whose specific gravity is to that of water :: m : 1, is formed into a hollow cone, and immersed with its vertex downward; it is required to find the ratio of the exterior diameter of its base to the altitude, when the surface immersed is a minimum.

Let $r =$ the radius BC of the base, DC the altitude $= x$, DG the altitude to which it is immersed $= z$, $p = 3.14159$, &c. Then, by similar triangles, $x : r :: z : \dfrac{rz}{x} = GF$; and the content of the conical part $EDF = p \times GF^2 \times \dfrac{GD}{3} = \dfrac{p r^2 z^3}{3 x^2} =$ the bulk of water displaced; and the weight of the water displaced, or $\dfrac{p r^2 z^3}{3 x^2} \times 1, = m \times 1$, the weight of the cubic inch of metal;

$\therefore z = \left(\dfrac{3 m}{p r^2}\right)^{\frac{1}{3}} \times x^{\frac{2}{3}}$. Also, since the surface of a cone $= \frac{1}{2}$ the circumference of the base multiplied into the slant side, the surface of $ADC = pr \cdot (r^2 + x^2)^{\frac{1}{2}}$; but similar surfaces are as the squares of corresponding sides;

$\therefore x^2 : z^2 :: pr \cdot (r^2 + x^2)^{\frac{1}{2}} : \dfrac{prz^2 \cdot (r^2 + x^2)^{\frac{1}{2}}}{x^2}$, the surface

immersed; which, by substitution, $= \dfrac{p\,r\,.\,(r^2 + x^2)^{\frac{1}{2}}}{x^2} \times$

$\dfrac{(3\,m)^{\frac{2}{3}}}{p\,r^2} \times x^{\frac{4}{3}}$, a minimum by the Problem; or $\dfrac{(r^2 + x^2)^{\frac{1}{2}}}{x^{\frac{2}{3}}}$

is a minimum; and the fluxion being put $= 0$, $x = r\,2^{\frac{1}{2}}$.

PROB. 39.

A cylinder of oak is depressed in water till its top is just level with the surface, and then is suffered to ascend ; it is required to determine the greatest altitude to which it will rise, and the time of its ascent.

Let $h =$ the height, and a the base of the cylinder, and suppose the specific gravity of oak : that of water :: n : 1. Let x be any variable altitude through which the cylinder has ascended, and $l = 16\frac{1}{12}$ feet. Then the moving force by which the cylinder endeavours to descend $= h \times a \times n$, and the force of the water upwards to prevent it $= (h - \ddot{x})$ $\times a \times 1$; $\therefore$ the whole moving force upon the cylinder $=$ $(h - x) \times a - h \times a \times n = ah - ax - h \times a \times n = (1 - n)\,.$ $ha - ax = mha - ax$, by substituting m for $1 - n$, $=\; a\,.$ $(mh - x)$. Hence the accelerating force $= \dfrac{a\,.\,(mh - x)}{h \times a \times n} =$ $\dfrac{mh - x}{nh}$. Now if v represent the velocity of the cylinder after it has risen through a space $= x$, $v\dot{v} = \pm 2\,lF\dot{x} =$, in this case, $2\,l \times \dfrac{mh\dot{x} - x\dot{x}}{nh}$; $\therefore v^2 = 2\,l \times \dfrac{2mhx - x^2}{nh}$, and

$v = \left(\dfrac{2\,l}{n\,h}\right)^{\frac{1}{2}} \cdot (2\,mhx - x^2)^{\frac{1}{2}}$. And when the cylinder has acquired its greatest ascent, $v = 0$, or $(2hmx - x^2)^{\frac{1}{2}} = 0$; $\therefore x = 2\,mh =$ the part of the cylinder extant.

To find the greatest velocity. Assume $2mhx - x^2$ a maximum; then $2mh\dot{x} = 2x\dot{x}$, and $x = mh =$ half the part of the cylinder extant, when it has acquired its greatest ascent. The velocity of the cylinder both in its ascent and descent increases to the middle point, and then decreases till it becomes $= 0$; so that the several vibrations are completed in equal times, and would be performed through the same space, were it not for the resistance of the air, the tenacity, friction, &c. of the water. When the cylinder becomes quiescent, the length of the part extant $= mh$. The motion is analogous to that of a pendulum.

To find the time, we have $\dot{T} = \dfrac{\dot{x}}{v} = \left(\dfrac{mh}{2l}\right)^{\frac{1}{2}} \times$

$\dfrac{\dot{x}}{(2mhx - x^2)^{\frac{1}{2}}}$; and $T = \left(\dfrac{nh}{2l}\right)^{\frac{1}{2}} \times A$, a circular arc of

radius 1, and versed sine $\dfrac{x}{mh}$. Now when the cylinder

has made half a vibration, $x = mh$; or the versed sine $= 1$;

$\therefore$ the time of half a vibration $= \left(\dfrac{nh}{2l}\right)^{\frac{1}{2}} \cdot Q$, the arc of a

quadrant; and the whole time of a vibration $= \left(\dfrac{2nh}{l}\right)^{\frac{1}{2}}$.

$Q = \dfrac{8}{15} \cdot h^{\frac{1}{2}}$. Suppose the time to be one second, $1 =$

$\dfrac{8}{15} \times h^{\frac{1}{2}}$, and $h = \dfrac{15^2}{8^2}$ nearly $=$ about 42 inches; that is,

a cylinder of oak, whose length is about 42 inches, would make each vibration in 1″. See Dr. Hutton's *Course.* Vol. III.

PROB. 40.

Let Q be an object placed beyond the principal focus F of a convex lens; to find its position when its distance Qq from its image q is a minimum.

Let $QE=x$, $FE=a$;
then $QF : QE :: QE : Qq$,
or $x-a : x :: x : Qq=$

$\dfrac{x^2}{x-a}$, which is a minimum;

$\therefore 2x\dot{x}.(x-a)-x^2\dot{x}=0$; that is, $2x^2\dot{x}-2ax\dot{x}-x^2\dot{x}=0$;
$\therefore x=2a$, or $QE=2EF$, and $QF=FE$.

PROB. 41.

If a person view himself in a concave reflector, his image decreases from the reflector to the principal focus, and then increases in going from it.

Let E be the center, and T the principle focus of the concave .reflector AC. Let pq be the image of PQ; the ap-

parent magnitude of PQ to an eye situated at Q is $\dfrac{pq}{Qq}$. Take $ET=TC=a$, $QC=x$, $PQ=M$; then $TQ :$

$TE :: TE : Tq = \dfrac{TE^2}{TQ} = \dfrac{a^2}{a-x}$; $\therefore Cq = Tq - TC =$

$\dfrac{a^2}{a-x} - a = \dfrac{ax}{a-x}$, and $Qq = x + \dfrac{ax}{a-x} = \dfrac{2ax-x^2}{a-x}$. Also

$(M)\ PQ : pq :: QC : qC :: x : \dfrac{ax}{a-x} :: a-x : a;$

$\therefore pq = \dfrac{Ma}{a-x}$; $\therefore$ the apparent magnitude or $\dfrac{pq}{Qq} \propto \dfrac{Ma}{a-x} \times$

$\dfrac{a-x}{2ax-x^2} \propto \dfrac{1}{2ax-x^2}$, whose fluxion $= \dfrac{2x\dot{x}-2a\dot{x}}{(2ax-x^2)^2} =$

$\dfrac{2\dot{x}.(x-a)}{(2ax-x^2)^2}$, which is negative, while x is less than a,

or QC less than TC; hence the apparent magnitude is at that time decreasing. But when x is greater than a, or QC greater than TC, the expression for the fluxion of the apparent magnitude becomes positive, and the magnitude of the image increases.

<div align="center">PROB. 42.</div>

If the eye and an object be both fixed, and a concave lens be moved from the object toward the eye, the apparent magnitude of the object will decrease to the middle point, and then begin to increase. Required a proof.

Let $ABDC$ be the lens, O the place of the eye, F the principal focus of rays coming in a contrary direction, PQ the object, and pq the image. The angle, under which pq
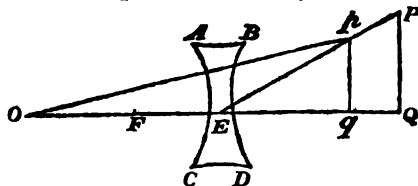
appears to the eye at O, varies as $\dfrac{pq}{qO}$. Assume $QO = 2a$,

$FE = p$, $QE = x$, and $PQ = M$.

Then, since $QF : FE :: QE : Eq$, $Eq = \dfrac{QE \times FE}{QF}$

$= \dfrac{px}{p+x}$. Also $QF : QE :: QE : Qq$; $\therefore Qq =$

$\dfrac{QE^2}{QF} = \dfrac{x^2}{p+x}$; hence $Oq = OQ - Qq = 2a - \dfrac{x^2}{p+x} =$

$\dfrac{2ap + 2ax - x^2}{p+x}$. And $PQ\ (M) : pq :: QE : qE :: x :$

$\dfrac{px}{p+x}$; $\therefore pq = \dfrac{M \times p}{p+x}$. Hence the apparent magnitude,

or the angle qop, which varies as $\dfrac{pq}{qO}$, varies as $\dfrac{Mp}{p+x} \times$

$\dfrac{p+x}{2ap+2ax-x^2}$, $\propto \dfrac{1}{2ap+2ax-x^2}$. The fluxion of this

quantity is $\dfrac{2\dot{x} \cdot (x-a)}{(2ap+2ax-x^2)^2}$, which is negative, while

x is less than a, but positive, when x becomes greater;
that is, the apparent magnitude decreases to the middle
point, and afterwards increases.

PROB. 43.

To find the position of Venus, when brightest.

Let E be the Earth, S the Sun, and V Venus; join
SV, SE, EV, and produce EV
to A, making $VA = VS$; with V
as a center, and VS radius, describe
the circular arc SA, and draw SB
perpendicular to EA. Then SEV
is the angle of elongation, SVA
the exterior angle, VB its cosine,
and BA its versed sine to the
radius SV. Take $SE = a$, $EV = x$,
$VB = y$, $SV = b$. Then the visible
illumined part $\propto BA \propto b - y$; and

the brightness $\propto \dfrac{b-y}{x^2}$, $\propto \dfrac{b}{x^2} - \dfrac{y}{x^2}$,
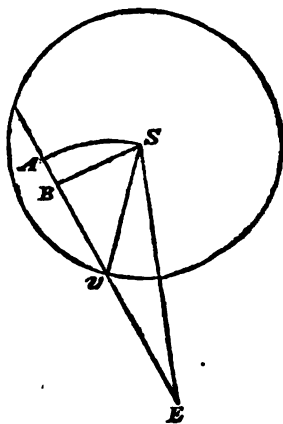
which by the Problem is a maximum. Now $SE^2 =$
$EV^2 + SV^2 + 2EV \times VB$, or $a^2 = x^2 + b^2 + 2x \times y$;

$\therefore y = \dfrac{a^2 - b^2 - x^2}{2x} =$ (if $m^2 = a^2 - b^2$) $\dfrac{m^2 - x^2}{2x}$; $\therefore$ by sub-

stituting for y, we have $\dfrac{b}{x^2} - \dfrac{m^2 - x^2}{2x^3}$ a maximum;

that is, $\dfrac{2bx - m^2 + x^2}{2x^3}$ is a maximum; hence $(2b\dot{x} + 2x\dot{x}) \times$

$2x^3 - 6x^2\dot{x} \times (2bx - m^2 + x^2) = 0$; or if we divide by $2x^2\dot{x}$, it may be reduced to this form; $-x^2 - 4bx + 3m^2 = 0$; $\therefore \ x^2 + 4bx = 3m^2$, from which equation $x = -2b + (4b^2 + 3m^2)^{\frac{1}{2}} = -2b + (3a^2 + b^2)^{\frac{1}{2}}$. Hence the three sides of the triangle SEV are known, to find the angle of elongation SEV, which is equal to $39°.44'$.

<center>PROB. 44.</center>

To find the position of Mars, when least bright.

Let E represent the Earth, S the Sun, M Mars at the time required. Join SE, SM, EM; and suppose EM to be produced to D, making $MD = MS$. With M as a center and MS radius describe the circle ASF; draw SB perpendicular to DE. Then SEM is the angle of elongation, and SMD
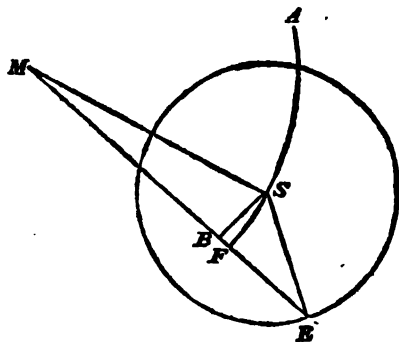
the exterior angle of elongation, whose versed sine is DB. Take $ES = a$, $MS = b$, $EM = x$, $MB = y$; then the brightness $\propto \dfrac{b+y}{x^2}$. But $ES^2 = EM^2 + MS^2 - 2EM \times MB$, or $a^2 = x^2 + b^2 - 2xy$; $\therefore y = \dfrac{b^2 - a^2 + 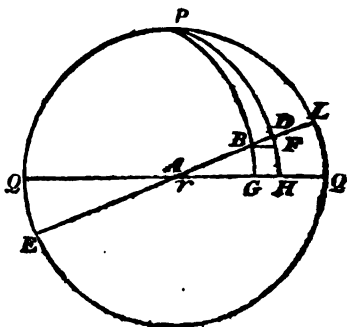x^2}{2x} = \dfrac{m^2 + x^2}{2x}$, if $m^2 = b^2 - a^2$; $\therefore$ the brightness $\propto \dfrac{b}{x^2} + \dfrac{m^2 + x^2}{2x^3} \propto \dfrac{2bx + m^2 + x^2}{2x^3}$, which is a minimum. And the fluxion $(2b\dot{x} + 2x\dot{x}) \times 2x^3 - 6x^2\dot{x} \times (2bx + m^2 + x^2) = 0$, from which equation x may be found $= -2b + (4b^2 - 3m^2)^{\frac{1}{2}} = -2b + (b^2 + 3a^2)^{\frac{1}{2}}$.

<center>C C</center>

PROB. 45.

The Sun being supposed to move uniformly in the ecliptic; to find when that part of the equation of time, which arises from the obliquity of the ecliptic, is a maximum.

Let QQ represent the equinoctial, EL the ecliptic, A the first point of Aries, PLQ the solstitial colure. Then at A the Sun's longitude and right ascension are equal; and they are again equal when the Sun is at L; but at any intermediate point, as B, PBG being a declination circle, the longitude AB is greater than the right ascension AG; it appears, therefore, that from A to a certain point the longitude increases faster than the right ascension, and from that point to L the right ascension increases faster than the longitude; hence the equation is a maximum, when the daily increment of right ascension is equal to the daily increment of longitude. At that time let the Sun be at B, and let BD be the increment of longitude, and GH of right ascension, in one day. Draw the circle of declination PDH, and the small arc BF parallel to GH.

Now $GH : BF :: r : \cos. BG$,

and $BF : BD :: S, BDF : r$ (since the triangle BDF is extremely small) ;

$\therefore GH : BD :: S, BDF : \cos. BG :: S, ABG : \cos. BG$, and $GH = BD$ by hypoth. ; $\therefore S, ABG = \cos. BG$.
But by Napier's Rules, $r \times \cos. \angle BAG = S, ABG \times \cos. BG$; that is, $r \times \cos. BAG = (\cos.)^2 BG$; hence r :

cos. BG :: cos. BG : cos. BAG; or the equation is a maximum, when the cosine of declination is a mean proportional between radius and the cosine of the obliquity of the ecliptic. The longitude at that time $= 46°. 14'$.

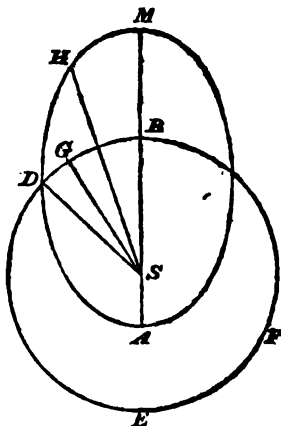Or thus, Let $AB = l$; $AG = a$; $BD = \dot{l}$; $GH = \dot{a}$;

Then $\dot{l} : \dot{a} :: S, 2AB : S, 2AG$, (Chap. xxiv. Prop. 12. Cor.) $\therefore$ in the case proposed, $S, 2AB = S, 2AG$; $\therefore 2AB$ is the supplement of $2AG$, and $AB + AG = 90°$.

PROB. 46.

To find when that part of the equation of time is a maximum, which arises from the unequal angular motion of the Sun in the ecliptic.

Let MDA represent the ecliptic, MSA its major axis, and S the Earth in one of its foci. With S as a center, and SD a mean proportional between the semi-axes of ADM as radius, describe the circle DEF. The area of this circle is equal to that of the ellipse; $\therefore$ if a body be conceived to revolve in this circle with the Sun's mean angular velocity, its periodic time will equal that of the Sun in the ecliptic; for the areas described in the two cases *dato tempore* are the same. Let this imaginary body be conceived to set off from B at the same time that the Sun begins its motion at the higher apse at M. The Sun's velocity at M is less than the mean; therefore the angle BSG described by the body in some given time is greater than BSH described by the Sun in the same time; and their difference, or the equation, will continue to increase

till the angular velocity of the Sun is equal to that of the body; hence the equation at that point is a maximum. Now the angular velocity varies as the area described in a given time directly, and the square of the distance inversely; ∴ since the area described in a given time is the same in both cases, when the angular velocities are equal, the distances are equal; or the equation is a maximum when the Sun is at *D*.

The absolute equation from both these causes is a maximum about the first of November.

PROB. 47.

To a pendulum *SA* of a given length, suspended at *S*, a given weight *n* is affixed at *A*; to find where another weight must be fixed, so that it may vibrate in the least time possible.

Let *F* be the point required, and *O* the center of oscillation; then the pendulum itself being considered as of no weight, if $SA = a$, and $SF = x$, $SO = \dfrac{na^2 + mx^2}{na + mx}$ (Art. 71.) Now the time of oscillation $\propto (SO)^{\frac{1}{2}}$; ∴ since the time is a minimum, $\dfrac{na^2 + mx^2}{na + mx}$ is a minimum; and its fluxion, that is, $2mx\dot{x} \cdot (na + mx) - m\dot{x} \times (na^2 + mx^2) = 0$, or $2nax + 2mx^2 = na^2 + mx^2$; ∴ $x^2 + \dfrac{2nax}{m} = \dfrac{na^2}{m}$; and from this quadratic

$$x = \frac{a}{m} \cdot (n^2 + mn)^{\frac{1}{2}} - \frac{na}{m} = SF.$$

PROB. 48.

Let *AB* represent a straight lever moveable round an horizontal axis of motion, which passes through *S*; suppose a weight *q* to be affixed to the extremity of the

shorter arm SA, and the power p at the extremity of the longer SB; Required the ratio of the arms, when the effect of p to turn the system at the first instant of motion is a maximum, the inertia of the lever not being considered.

Let $SA=x$, $SB=a$. The moving force of $p=p$; but the weight q would balance a power

$$A \overline{ \overset{S}{\wedge} } B$$

at $B=\dfrac{q \times AS}{BS}=\dfrac{q x}{a}$; $\therefore$ the moving

force acting on $B=p-\dfrac{q x}{a}=\dfrac{p a - q x}{a}$.

Also the inertia or mass is obtained by supposing the bodies p and q to be removed, and a weight $= p + \dfrac{q x^2}{a^2}$, or $\dfrac{p a^2 + q x^2}{a^2}$ placed at B (Art. 65.); hence the force, which accelerates B at the first instant of motion, $= \dfrac{(p a - q x) . a}{p a^2 + q x^2}$, which is in this case a maximum;

$$\therefore \quad - q \dot{x} . (p a^2 + q x^2) - 2 q x \dot{x} . (p a - q x) = 0,$$
$$\text{or } p a^2 + q x^2 + 2 p a x - 2 q x^2 = 0;$$

and by solving the quadratic, $x = \dfrac{p a + (p^2 a^2 + p q a^2)^{\frac{1}{2}}}{q}$.

PROB. 49.

Let the arms of the lever ASB be given, and a given power p be affixed at B; required the weight y suspended at A, so that its momentum in a small given time may be a maximum, the inertia of the lever not being considered. (See preceding Fig.)

Let $SB=a$, $SA=b$; then the moving force at B, as before, $=p -\dfrac{y \times b}{a}=\dfrac{p a - y b}{a}$; and the inertia $=p + y \times \dfrac{b^2}{a^2}=$

$$= \frac{p\,a^2 + y b^2}{a^2}\; ; \; \therefore \text{ the accelerating force upon } B \text{ at first} =$$

$\frac{pa - yb}{pa^2 + yb^2} \times a.$ Now the force which accelerates B : that

which accelerates A :: SB : SA :: a : b; $\therefore$ the force

which accelerates $A = \frac{pa - yb}{pa^2 + yb^2} \times b$; and this varies as

the velocity; hence, at the commencement of the motion,

the momentum $\propto \frac{p\,ay - by^2}{p\,a^2 + b^2 y}$, a maximum; $\therefore (pa\dot{y} - 2by\dot{y})$

$: (pa^2 + b^2 y) - b^2 \dot{y} \cdot (p\,ay - b\,y^2) = 0$; hence, dividing

by $\dot{y}$, and multiplying the quantities, $b^2 y^2 + 2pa^2 by = p^2 a^2$,

and $y = \frac{pa}{b^2} \cdot (ab + a^2) - \frac{p\,a^2}{b^2}$.

PROB. 50.

Given the radii SA, SB of a wheel and axle, and let
a given weight p, applied at the circumference of the
wheel, raise a weight y applied at the circumference of
the axle; to find y, when the momentum communicated
to it in a given time is a maximum, the inertia of the
wheel and axle not being considered.

Let $SA = a$, $SB = b$; then the moving force upon

$A = p - \frac{yb}{a}$; and the mass sup-

posed to be collected at $A = p + $

$\frac{y b^2}{a^2}$; therefore the accelerating

force at $A = \frac{pa - yb}{pa^2 + yb^2} \times a$; hence

the accelerating force at $B =$

$\frac{pa - yb}{pa^2 + yb^2} \times b$; $\therefore$ the momentum communicated to $y \propto$
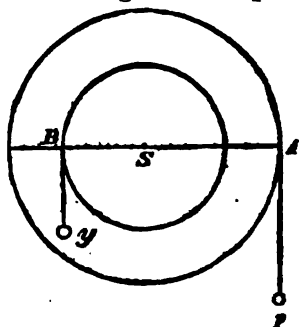
$\dfrac{p\,a\,y - y^2 b}{p\,a^2 + y\,b^2}$, a maximum; and the fluxion $= 0$, or

$(p\,a\,\dot{y} - 2b\,y\dot{y}) \cdot (p\,a^2 + y\,b^2 - b^2\dot{y} \cdot (p\,a\,y - y^2 b) = 0$; whence, dividing by $\dot{y}$, and multiplying the quantities $b^2 y^2 + 2\,p\,a^2 b\,y = p^2\,a^3$, the same equation as in the last case;

and $y = \dfrac{p\,a}{b^2} \cdot (a\,b + a^2) - \dfrac{p\,a^2}{b^2}$.

PROB. 51.

Given two weights p and q, acting, as in the former case, at A and B, and the radius SB of the axle; to find the radius of the wheel, so that p may draw up q through a given space in the least time possible, the inertia of the wheel and axle not being considered.

Let $SB = b$, $SA = x$; then the accelerating force upon q, by the preceding Problem, $= \dfrac{(p\,x - q\,b) \cdot b}{p\,x^2 + q\,b^2}$. Now $S \propto$

$F \times T^2$; therefore $T^2 \propto \dfrac{1}{F}$, when S is given; hence, since the time is a minimum, the reciprocal of the force,

or $\dfrac{p\,x^2 + q\,b^2}{(p\,x - q\,b) \cdot b}$, is a minimum; $\therefore$ its fluxion $= 0$;

Hence $2\,p\,x\dot{x} \cdot (p\,x - q\,b) - p\,\dot{x} \cdot (p\,x^2 + q\,b^2) = 0$;

$\therefore 2\,p\,x^2 - 2\,q\,b\,x - p\,x^2 - q\,b^2 = 0$,
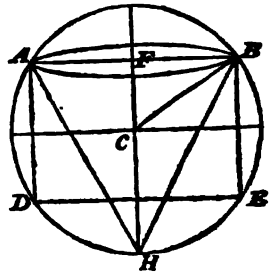
or $p\,x^2 - 2\,q\,b\,x = q\,b^2$,

from the solution of which quadratic $x = \dfrac{q\,b + (q^2 b^2 + p\,q\,b^2)^{\frac{1}{2}}}{p}$.

SECTION II.

PROB. 52.

To inscribe the greatest cylinder in a given sphere.

Let $ABED$ be the cylinder; $CB=r$, $CF=x$, $BF=y$. Then $y^2=(r^2-x^2)$; and the content$=2p.$ (r^2x-x^3), a maximum; $\therefore r^2\dot{x}=$ $3x^2\dot{x}$, and $x=\dfrac{r}{3^{\frac{1}{2}}}$.

PROB. 53.

To inscribe the greatest cone in a given sphere.

Let HAB be the cone required. Then, since the content of a cone $=\frac{1}{3}$ of a cylinder of the same base and altitude, (Art. 51. Ex. 6.) the content of $HAB = p \times FB^2 \times \dfrac{HF}{3}$. Let $CH=r$, $HF=x$; then $BF^2 = 2rx - x^2$; therefore $p \times \dfrac{(2rx-x^2)\,.\,x}{3}$ is a maximum; hence $2rx^2-x^3$ is a maximum, and $4rx\dot{x}=3x^2\dot{x}$; $\therefore x = \dfrac{4r}{3}$.

PROB. 54.

To inscribe the greatest cylinder in a given spheroid, the axis of the cylinder being supposed to coincide with the axis of the spheroid.

Let $a =$ the semi-major axis, b the semi-minor, y the ordinate, and x that part of the axis intercepted between the ordinate and the center. Then $y^2 = \dfrac{b^2}{a^2} \cdot (a^2 - x^2)$;

$\therefore p y^2 \times 2 x$, the content, $= \dfrac{2 p b^2}{a^2} \cdot (a^2 x - x^3)$, a maximum;

$\therefore a^2 \dot{x} = 3 x^2 \dot{x}$, and $x = \dfrac{a}{3^{\frac{1}{2}}}$.

PROB. 55.

To inscribe the greatest cone in a given spheroid, its vertex coinciding with the extremity of the major axis.

Let the part of the axis intercepted between the vertex of the cone and its base $= x$; then $y^2 = \dfrac{b^2}{a^2} \cdot (2 a x - x^2)$;

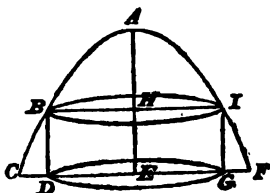and the content of the cone $= \dfrac{p b^2}{3 a^2} \cdot (2 a x^2 - x^3)$, a maximum; therefore $4 a x \dot{x} = 3 x^2 \dot{x}$, and $x = \dfrac{4 a}{3}$.

PROB. 56.

To inscribe the greatest cylinder in a given paraboloid CAF.

Let $BIGD$ be the cylinder; $AH = x$, $AE = b$, $HI = y$, and the *latus rectum* $= c$. Then $y^2 = cx$; $\therefore$ the content, or $p y^2 \times HE, = pcx . (b - x)$, a maximum; hence $b \dot{x} = 2 x \dot{x}$, and $x = \dfrac{b}{2}$.
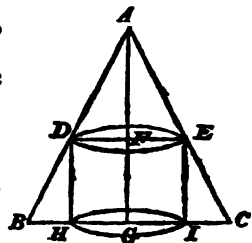
### PROB. 57.

To inscribe the greatest cylinder in a given cone.

Let $AF = x$, $FE = y$, $AG = a$, $GB = b$; then $y = \dfrac{b\,x}{a}$; therefore the content of the cylinder, or $py^2 \times FG = \dfrac{p\,b^2}{a^2} \times x^2 \cdot (a - x)$, a maximum; hence $ax^2 - x^3$ is a maximum, and $2\,a\,x\,\dot{x} = 3\,x^2\,\dot{x}$, or $x = \dfrac{2\,a}{3}$.

PROB. 58.

Find that point in the side of the triangle ABC, from which, if perpendiculars be drawn to the other two sides, their product may be a maximum.

Let D be the point required; draw DE perpendicular to CB, and DF to AB; also from A and C draw AL and CG perpendicular to CB and AB. Then by similar triangles, ADF, ACG, and CDE, CAL,
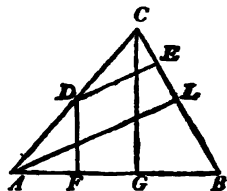
$$AD : DF :: AC : CG$$
$$CD : DE :: CA : AL$$

$$\therefore AD \times DC : DF \times DE :: AC^2 : CG \times AL.$$

Now as the last two terms are constant in the same triangle, $DF \times DE \propto AD \times DC$, and is therefore a maximum when $AD \times DC$ is a maximum; that is, when $AD = DC$ (Art. 23. Ex. 2.); hence AC must be bisected in D, and D is the point required.

This Figure answers for an acute-angled triangle; the

process is nearly the same for a right and obtuse-angled triangle.

PROB. 59.

From a given cone ABC, to cut the greatest parabola DEF.

Let BC be that diameter of the base which is perpendicular to EF. Take $CG=x$, $CB=b$, $BA=a$; then, by the property of the circle, $EG = (BG \times GC)^{\frac{1}{2}} = ((b-x).x)^{\frac{1}{2}} = (bx-x^2)^{\frac{1}{2}}$; $\therefore EF = 2.(bx - x^2)^{\frac{1}{2}}$. Also $CB : BA :: CG : GD$, or $b : a :: x : GD = \frac{ax}{b}$; $\therefore$ the area of the parabola
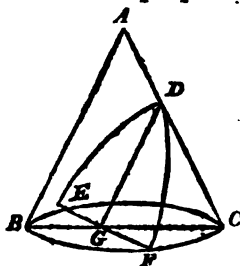
$$EDF = \frac{2}{3} \times \frac{ax}{b} \times 2.(bx - x^2)^{\frac{1}{2}} \quad \text{(Art. 48. Ex. 2.)}, \quad \text{which}$$

varies as $x.(bx-x^2)^{\frac{1}{2}}$, a maximum; hence its square $bx^3 - x^4$ is a maximum, and its fluxion $=0$; $\therefore 3bx^2\dot{x} = 4x^3\dot{x}$,

and $x = \dfrac{3b}{4}$.

PROB. 60.

The distance of the center of gravity from the vertex of a solid, formed by the revolution of a superficies of the parabolic kind, is $\frac{7}{8}$ of its axis; required the nature of the generating curve.

The fluent of $\dfrac{y^2 x \dot{x}}{f. y^2 \dot{x}} = \dfrac{7}{8} x$. (Art. 61.)

Assume an equation $a^{n-1}x = y^n$; then $y^2 = a^{\frac{n-2}{n}} x^{\frac{2}{n}}$, and

$$\frac{f.\,y^2 x \dot{x}}{f.\,y^2 \dot{x}} = \frac{f.\,a^{\frac{2n-2}{n}} x^{\frac{2}{n}+1}\,\dot{x}}{f.\,a^{\frac{2n-2}{n}} x^{\frac{2}{n}}\,\dot{x}} = \frac{\dfrac{(n+2)}{n} \times x^{\frac{2n+2}{n}}}{\dfrac{2n+2}{n} \times x^{\frac{n+2}{n}}} = \frac{n+2}{2n+2} \times x,$$

or $\dfrac{n+2}{2n+2} = \dfrac{7}{8}$; $\therefore\, 8n + 16 = 14n + 14$, and $6n = 2$, and $n = \dfrac{1}{3}$;

$\therefore$ the equation is $x = a^{\frac{2}{3}} y^{\frac{1}{3}}$, or $x^3 = a^2 y$.

In the common parabola, $\dfrac{n+2}{2n+2} = \dfrac{2}{3}$; $\therefore\, 3n + 6 =$ $4n + 4$, and $n = 2$; $\therefore\, a x = y^2$, the equation.

PROB. 61.

Required to find the nature of the curve, in which the sub-tangent : the sub-normal :: $m^2 x^2$: y^2.

By the Problem, $\dfrac{y \dot{x}}{\dot{y}} : \dfrac{y \dot{y}}{\dot{x}} :: m^2 x^2 : y^2$;

$\therefore\, \dot{x}^2 : \dot{y}^2 :: m^2 x^2 : y^2$, and $\dot{x} : \dot{y} :: m x : y$;

$\therefore\, \dfrac{\dot{x}}{x} = \dfrac{m \dot{y}}{y}$, and hyp. log. $x = m \times$ hyp. log. y; $\therefore\, x \propto y^m$,

and $a^{m-1} x = y^m$ the equation.

PROB. 62.

Required the equation to a curve, whose sub-tangent $= n$ times its abscissa.

Here $\dfrac{y \dot{x}}{\dot{y}} = n x$; $\therefore\, \dfrac{\dot{x}}{x} = n \times \dfrac{\dot{y}}{y}$, and hyp. log. $x = n \times$ hyp. log. y; $\therefore\, x \propto y^n$, or $a^{n-1} x = y^n$.

If $n = 2$, $a x = y^2$, the equation to the parabola.

PROB. 63.

To find the nature of the curve, whose tangent is a given quantity.

Let $x =$ the abscissa, and y the ordinate; let the tangent $= b$; then, since the sub-tangent $= \dfrac{y\dot{x}}{\dot{y}}$, we have
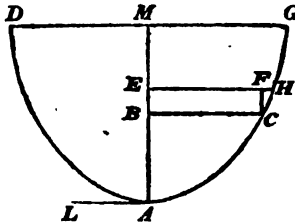
$$\frac{y^2 \dot{x}^2}{\dot{y}^2} + y^2 = b^2; \quad \therefore y^2 . (\dot{x}^2 + \dot{y}^2) = b^2 \dot{y}^2, \quad \text{or} \quad y^2 \dot{z}^2 = b^2 \dot{y}^2; \quad \therefore$$

$\dot{z} = \dfrac{b\dot{y}}{y}$, and $z \propto b \times$ hyp. log. y; or $az = b \times$ hyp. log. y, the equation.

Prob. 64.

To determine the law of the weights, which press upon each particle of a perfectly flexible line, so that it shall form a curve whose equation is $a^3 x = y^4$.

Let DAG represent the curve, AM its axis, and A the lowest point. Draw EH, BC, two ordinates to the axis, indefinitely near each other; let CF be drawn perpendicular to EH, and AL be a tangent at A. Then AC being considered as inflexible, after it has assumed the proper situation, it is kept at rest by three forces; at A by the action of AD in the direction AL; at C by the part of the line CG in a tangential direction, and by the pressure in the direction FC; hence, if the effect of $AD=b$, $AB=x$, $BC=y$,

$$\dot{y} : \dot{x} :: b : \text{pressure}; \quad \therefore \dot{x} = \frac{Pr \times \dot{y}}{b};$$

but by the nature of the curve, since $a^3 x = y^4$, $\dot{x} = \dfrac{4y^3 \dot{y}}{a^3}$;

$$\therefore \frac{Pr \times \dot{y}}{b} = \frac{4y^3 \dot{y}}{a^3}, \quad \text{and the pressure} \propto y^3.$$

PROB. 65.

Let AP be the abscissa of any curve, $PMNQ$ an ordinate revolving about the fixed point P, and cutting the curve in as many points as it has dimensions; My, Nx, and Qw, being drawn tangents to the curve in M, N, and Q; it is required to find the sum of the reciprocal

sub-tangents $\dfrac{1}{Py} + \dfrac{1}{Px} + \dfrac{1}{Pw}$.

Let the equation to the curve be $y^n - (a'x + b') . y^{n-1} + (c'x^2 + d'x + e') . y^{n-2} - \&c. + px^n - qx^{n-1} + rx^{n-2} - \&c. = 0$; let a, b, c be the values of y corresponding to AP or x, the abscissa;

then, since the last term is the product of all the roots with their signs changed, $a \times$
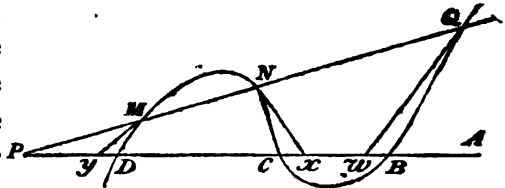
$b \times c \times \&c. = px^n - qx^{n-1} + \&c.$; hence $\dot{a}bc, \&c. + \dot{b}ac, \&c. + \dot{c}ab, \&c. = npx^{n-1}\dot{x} - (n-1).qx^{n-2}\dot{x} + \&c.$; $\therefore$ dividing the former part of the equation by $a \times b \times c \times \&c.$ and the latter by $px^n - qx^{n-1} + \&c.$ we have,

$$\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} + \&c. = \frac{npx^{n-1}\dot{x} - (n-1).qx^{n-2}\dot{x} + \&c.}{px^n - qx^{n-1} + \&c.} ;$$

hence, $\dfrac{\dot{a}}{a\dot{x}} + \dfrac{\dot{b}}{b\dot{x}} + \dfrac{\dot{c}}{c\dot{x}} + \&c. = \dfrac{npx^{n-1} - (n-1).qx^{n-2} + \&c.}{px^n - qx^{n-1} + \&c.}$

that is, $\dfrac{1}{Py} + \dfrac{1}{Px} + \dfrac{1}{Pw} + \&c. = \dfrac{npx^{n-1} - (n-1).qx^{n-2} + \&c.}{px^n - qx^{n-1} + \&c.}$

$$= \frac{nx^{n-1} - (n-1) . \frac{q}{p}x^{n-2} + \&c.}{x^n - \frac{qx^{n-1}}{p} + \&c.} .$$

Cor. Since the roots of the equation $x^n - \frac{q}{p}x^{n-1} + \&c. = 0$, are AB, AC, AD, &c., the coefficients of this equation are constant; also x is constant, because P is a fixed point by the hypothesis; $\therefore$ $\dfrac{n x^{n-1} - (n-1).\frac{q}{p}x^{n-2} + \&c.}{x^n - \frac{q}{p}x^{n-1} + \&c.}$ is constant; that is, the sum of the reciprocal sub-tangents is a constant quantity.

Prob. 66.

In the same curve, to find the sum of the sub-normals.

If p and q be the coefficients of the second and third terms of an equation, the sum of the squares of the roots $= p^2 - 2q$; hence, on the supposition in the last Problem, $a^2 + b^2 + c^2 + \&c. = (a'x + b')^2 - (2c'x^2 + 2d'x + 2e')$. Take the fluxion; then $a\dot{a} + b\dot{b} + c\dot{c} = a'\dot{x}.(a'x + b') - (2c'x\dot{x} + d'\dot{x})$; $\therefore \dfrac{a\dot{a}}{\dot{x}} + \dfrac{b\dot{b}}{\dot{x}} + \dfrac{c\dot{c}}{\dot{x}} = a'^2 x + a'b - (2c'x + d')$;

that is, since the sub-normal $= \dfrac{y\dot{y}}{\dot{x}}$, where x is the abscissa and y the ordinate, in this case the sum of the sub-normals $= a'^2 x + a'b - (2c'x + d')$.

Prob. 67.

To draw a tangent to an ellipse, so that the triangle contained under this tangent, and the major and minor semi-axes produced, may be a minimum.

Let ABM be the ellipse, C the center. Let P be the point through which the tangent must be drawn; join CP; draw PN perpendicular to AC, and let the tangent at P meet the two axes produced in O and T.
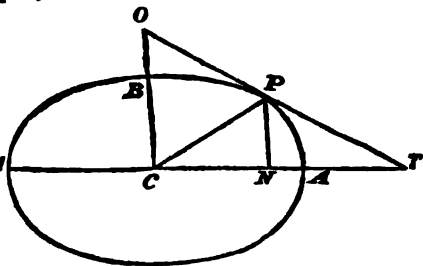
Take $AC=a$, $CB=b$, $CN=x$; then $PN=\dfrac{b}{a} \cdot (a^2 - x^2)^{\frac{1}{2}}$,

and $CT=\dfrac{a^2}{x}$ by conic sections; $\therefore NT=\dfrac{a^2}{x}-x=\dfrac{a^2-x^2}{x}$;

and by similar triangles, TNP, TCO, $TN : NP ::$

$TC : CO$, or $\dfrac{a^2-x^2}{x} : \dfrac{b}{a} \cdot (a^2-x^2)^{\frac{1}{2}} :: \dfrac{a^2}{x} : CO=\dfrac{ab}{(a^2-x^2)^{\frac{1}{2}}}$;

therefore $CO \times CT$, or $\dfrac{a^3 b}{x \cdot (a^2-x^2)^{\frac{1}{2}}}$, is a minimum;

hence $x \cdot (a^2 - x^2)^{\frac{1}{2}}$, or $a^2 x^2 - x^4$, is a maximum; $\therefore$

$2 a^2 x\dot{x} - 4 x^3 \dot{x}=0$, and $x =\dfrac{a}{2^{\frac{1}{2}}}$.

Cor. Since $x= \dfrac{a}{2^{\frac{1}{2}}}$, $CO\left(=\dfrac{ab}{(a^2-x^2)^{\frac{1}{2}}}\right)=b\times 2^{\frac{1}{2}}$; and

$NP \left(=\dfrac{b}{a}.(a^2 - x^2)^{\frac{1}{2}}\right)=\dfrac{b}{2^{\frac{1}{2}}}$; $\therefore NP : CO :: \dfrac{b}{2^{\frac{1}{2}}} : b \times 2^{\frac{1}{2}} ::$

$1 : 2$; $\therefore TP : TO :: 1 : 2$, or the tangent TO is bisected in P. The same is true for any oval figure.

PROB. 68.

The greatest parallelogram that can be inscribed in a curve ABC concave to its axis, and the least triangle

that can be described about it will be when the sub-tangent ET is equal to the base BE of the parallelogram, or half the base of the triangle.

By Art. 23, the greatest parallelogram which can be in-scribed in the triangle BTG, has its base $BE = \frac{1}{2} BT$. Now a greater parallelogram cannot be inscribed in the curve than in the circumscribing triangle; therefore $BEDK$ is the great-est parallelogram which can be inscribed in the curve.
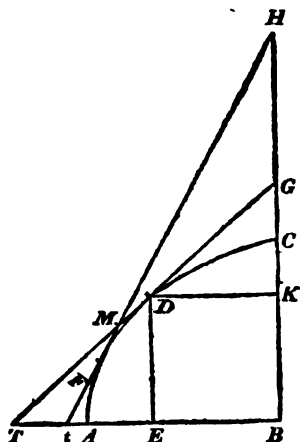
Also, if TG be bisected in D, where it touches the curve, BTG is the least triangle; if not, let BtH be less, and suppose tH to cut TG in M, and to touch the curve in F. Then, since $DG = DT$, MG is greater than MT; and Mt being less that MT, and consequently less than MG, must be much less than MH. Hence, since the vertical angles at M are equal, the triangle HMG, which we have added to the original triangle, is greater than TMt, which we have taken away; that is, the triangle BHt is greater than BGT.

The same kind of proof may be extended to any rectilinear figure, which circumscribes an oval; for the other sides being supposed constant, the figure is always the least, when the remaining side is bisected in the point of contact.

Prob. 69.

To find that point P in an ellipse, to which if a tangent be drawn, the part Py, intercepted between the

point P and the perpendicular Cy drawn from the center upon the tangent, may be a maximum.

Draw the conjugate diameter CD, and draw PF perpendicular to it. Let $Cy = p$, $PC = x$, $AC = a$, BC, the semi-minor, $= b$. Then $CD^2 + (CP^2) x^2 = a^2 + b^2$; ∴ $CD^2 = a^2 + b^2 - x^2$. Now $CD \times PF$, or $CD \times Cy$, $= a \times b$;

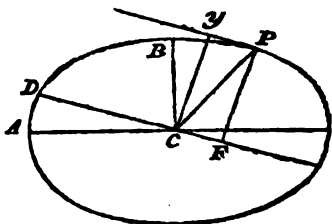∴ $CD^2 = \dfrac{a^2 b^2}{p^2}$; hence $\dfrac{a^2 b^2}{p^2} =$

$a^2 + b^2 - x^2$, and $p^2 = \dfrac{a^2 b^2}{a^2 + b^2 - x^2}$; ∴ $Py^2 = CP^2 - Cy^2$

$= x^2 - \dfrac{a^2 b^2}{a^2 + b^2 - x^2}$, whose fluxion $= 0$; ∴ $2x\dot{x} -$

$\dfrac{a^2 b^2 \times 2x\dot{x}}{(a^2 + b^2 - x^2)^2} = 0$; ∴ $(a^2 + b^2 - x^2)^2 = a^2 b^2$, and

$a^2 + b^2 - x^2 = ab$; that is, $CD^2 + x^2 - x^2 = AC \times CB$; ∴ $AC : CD :: CD : CB$; hence x is known, and therefore the point P.

PROB. 70.

To find the area of the parabola, considered as a spiral.

Let PQ represent the fluxion of the curve, Sy a perpendicular on the tangent at P; join SP, SQ, and draw QT perpendicular to SP. Then i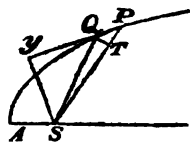f $SA = a$, and $SP = x$, $Sy = (ax)^{\frac{1}{2}}$, $PT = \dot{x}$, and SQP is the fluxion of the area SAQ. Now SPQ

$= \dfrac{SP \times QT}{2}$; and $QT : TP$, or $\dot{x} :: Sy$, or $(ax)^{\frac{1}{2}} : Py$,

or $(x^2 - ax)^{\frac{1}{2}}$; ∴ $QT = \dfrac{a^{\frac{1}{2}} x^{\frac{1}{2}} \dot{x}}{(x^2 - ax)^{\frac{1}{2}}} = \dfrac{a^{\frac{1}{2}} \dot{x}}{(x - a)^{\frac{1}{2}}}$; and the

fluxion of the area $= \dfrac{a^{\frac{1}{2}}x\dot{x}}{2\,(x-a)^{\frac{1}{2}}}$, and $\dot{f}. = \dfrac{a^{\frac{1}{2}}}{2} \times$:

$\dfrac{2x.(x-a)^{\frac{1}{2}}}{8} + \dfrac{4a.(x-a)^{\frac{1}{2}}}{3}$ (Fluent 19.) $=$ the area SAP;

the corr. $= 0$.

ON MERCATOR's PROJECTION OF THE SPHERE.

LEMMA.

The length of a degree of latitude at any place is to the length of a degree of longitude there as radius to the cosine of latitude.

Let P represent the pole, and C the center of the earth ; AB the equator, AB the length of a degree of longitude at the equator; PDA, PEB, two meridians ; from F draw in the planes PAC, PBC, the lines FD, FE parallel to AC, BC. The included arc DE is part of a small circle parallel to AB, and measures a degree in longitude at D. Now by similar sectors ACB, DFE; $AB : DE :: AC : DF$; or, since AB = the length of a degree of latitude at any place D, the length of a degree of latitude : the length of a degree of longitude :: radius : cosine of latitude.
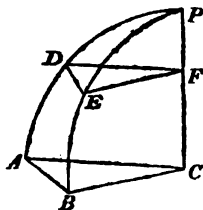
In Mercator's Projection, the sphere is projected upon a plane, P is at an infinite distance, and the meridians PA and PB are parallel. Hence, in all latitudes DE is the same ; therefore to preserve the just ratio between a degree of latitude and longitude in the projection, the degrees of latitude must increase in receding from the equator, according to the proportion in the Lemma.

Prob. 71.

In this projection, to find the length of an arc of the meridian corresponding to any given latitude.

Let PCp be the axis of the earth, AC the equator, D some point in the meridian PA; draw DF perpendicular to Pp, and join Dp, DC. Then, if $AC=r$, $AD=x$, $CF=y$, and the length of 'the projection of AD on the plane $=z$, we have, by the Lemma, $\dot{z} : \dot{x} :: r : (r^2 - y^2)^{\frac{1}{2}}$;
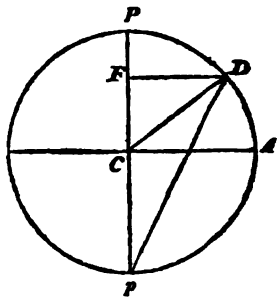
$\therefore \dot{z} = \dfrac{r\dot{x}}{(r^2 - y^2)^{\frac{1}{2}}}$. But (by Art. 44.)

$\dot{x} = \dfrac{r\dot{y}}{(r^2 - y^2)^{\frac{1}{2}}}$; $\quad \therefore \dot{z} = \dfrac{r^2 \dot{y}}{r^2 - y^2} = \dfrac{r}{2} \times \dfrac{2 r \dot{y}}{r^2 - y^2}$, and

$z = \dfrac{r}{2} \times$ hyp. log. $\dfrac{r + y}{r - y}$ (Art. 43.) $+$ corr. $= r \times$ hyp. log.

$\left(\dfrac{r + y}{r - y}\right)^{\frac{1}{2}} +$ corr. But $DF : Fp :: r :$ tan. of the angle

FDp, that is, $(r^2 - y^2)^{\frac{1}{2}} : r+y :: r : T, FDp$; $\therefore$ the

tangent of FDp, or co-tan. of FpD, or of $\frac{1}{2} DCP$,

the co-lat. $= r \times \left(\dfrac{r + y}{r - y}\right)^{\frac{1}{2}}$; hence $\left(\dfrac{r + y}{r - y}\right)^{\frac{1}{2}} =$

$\dfrac{\text{co-tan. of } \frac{1}{2} \text{ co-lat.}}{r}$, and $z = r \times$ hyp. log. $\dfrac{\text{co-tan of } \frac{1}{2} \text{ co-lat.}}{r}$

$+ C$. Now when $x = 0$, the co-tan. of $\frac{1}{2}$ the co-lat. is the co-tan. of $45°$, and equals r; hence $C = -r$

$\times$ hyp. log. $\dfrac{r}{r} = 0$; therefore $z = r \times$ hyp. log.

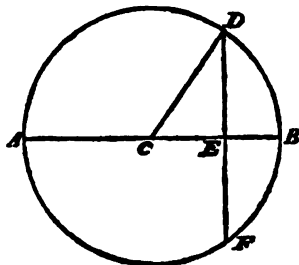$\dfrac{\text{co-tan. of } \frac{1}{2} \text{ co-lat.}}{r}$.

PROB. 72.

Given the arc of a circle, to find its sine and cosine.

Let BD the given arc $= z$, DE its sine $= y$, CE its cosine $= x$. Then, by the nature of the circle, $\dot{z} : -\dot{x} :: r : y$;

therefore $y = \dfrac{-r\dot{x}}{\dot{z}}$. Now to the cosine CE we have two values of z, BD and BF, which are equal to each other, but they must have different signs; if $z = a$, $z - a = 0$; if $z = -a$, $z + a = 0$, and the quadratic resulting from these two is $z^2 - a^2 = 0$. The same reasoning holds for any other corresponding values of z; hence the equation, whose roots are the several values of z, will contain only its even powers; also, if $CE = CB = r$, $z = 0$; therefore if CE be assumed in a series in terms of z, the first term will be r, and the succeeding terms will contain the even powers of z.

Let $x = r + az^2 + bz^4 + cz^6 + \&c.$

then $\dot{x} = 2az\dot{z} + 4bz^3\dot{z} + 6cz^5\dot{z} + \&c.$

$$\therefore y\left(= -\frac{r\dot{x}}{\dot{z}}\right) = -2arz - 4brz^3 - 6crz^5 - \&c.$$

and $\dot{y} = -2ar\dot{z} - 3.4brz^2\dot{z} - 5.6crz^4\dot{z} - \&c.$

But, in a circle, $\dot{z} : \dot{y} :: r : x$; $\therefore x\dot{z} - r\dot{y} = 0$;

hence $\left. \begin{array}{l} r\dot{z} + az^2\dot{z} + bz^4\dot{z} + \&c. \\ + 2ar^2\dot{z} + 3.4.br^2\dot{z} + 5.6.cr^2z^4\dot{z} + \&c. \end{array} \right\} = 0.$

Therefore, by equating the coefficients of the corresponding terms,

$$r + 2ar^2 = 0; \quad \therefore a = -\frac{1}{2r}.$$

$$a + 3.4br^2 = 0; \quad \therefore b = -\frac{a}{3.4.r^2} = \frac{1}{2.3.4r^3}.$$

$b + 5.6cr^2 = 0$; $\therefore c = -\dfrac{b}{5.6r^4} = -\dfrac{1}{2.3.4.5.6r^5}$, &c. ;

whence, by substituting for a, b, &c., their values in the assumed equation, $x = r - \dfrac{z^2}{2r} + \dfrac{z^4}{2.3.4r^3} - \dfrac{z^6}{2.3.4.5.6r^5} + $ &c.;

and $y = z - \dfrac{z^3}{2.3r^2} + \dfrac{z^5}{2.3.4.5r^4} - $ &c.

Prob. 73.

To sum the series $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + $ &c. ad inf.

In the last Problem, if $r = 1$, $y = z - \dfrac{z^3}{2.3} + \dfrac{z^5}{2.3.4.5} - $ &c.

let $y = 0$; then $z - \dfrac{z^3}{2.3} + \dfrac{z^5}{2.3.4.5} - $ &c. $= 0$, in which

one root $z = 0$; hence, dividing by z, $1 - \dfrac{z^2}{2.3} +$

$\dfrac{z^4}{2.3.4.5} - $ &c. $= 0$. Since $y = 0$, if $C = $ the semi-circumference, the other values of z are $1C$, $2C$, $3C$, &c. $-1C$, $-2C$, $-3C$, &c., each series ad inf. Let $v = \dfrac{1}{z}$; then $1 - \dfrac{1}{2.3.v^2} + \dfrac{1}{2.3.4.5.v^4} - $ &c. $= 0$; or, multiplying by v^n,

$$v^n - \dfrac{v^{n-2}}{2.3} + \dfrac{v^{n-4}}{2.3.4.5} - \text{ \&c.} = 0,$$

in which n values of $y = 0$; and the other values are $\dfrac{1}{1C}$, $\dfrac{1}{2C}$, $\dfrac{1}{3C}$, &c. ad inf., and $-\dfrac{1}{1C}$, $-\dfrac{1}{2C}$, $-\dfrac{1}{3C}$, &c. ad inf. Now the sum of the squares of the roots of this equation $= p^2 - 2q$; here, as the second term is wanting, $p = 0$, and $q = -\dfrac{1}{2.3}$; $\therefore p^2 - 2q = \dfrac{1}{3}$;

that is, since the squares of $\dfrac{1}{1C}$, $\dfrac{1}{2C}$, &c. are the same

as the squares of $-\dfrac{1}{1C}$, $-\dfrac{1}{2C}$, &c., $\dfrac{2}{1 \cdot C^2} + \dfrac{2}{2^2 C^2} +$

$\dfrac{2}{3^2 C^2} + $ &c. ad inf. $= \dfrac{1}{3}$; $\therefore \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + $ &c. ad inf.

$= \dfrac{C^2}{6}$.

Cor. 1. In the same manner the sums of any of the even powers of the reciprocals of the natural numbers may be found. Instead of finding the value of $p^2 - 2q$, take the algebraic expression required in that particular case. The sum of the reciprocals of the odd powers cannot be determined by this method, as the odd powers of the negative roots destroy the odd powers of the positive.

Cor. 2. Since $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + $ &c. $= \dfrac{C^2}{6}$,

$\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + $ &c. $+ \dfrac{1}{2^2} + \dfrac{1}{4^2} + $ &c. $= \dfrac{C^2}{6}$;

that is, $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + $ &c. $+ \dfrac{1}{2^2} \times \left(\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} \right) + $ &c. $= \dfrac{C^2}{6}$,

or $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + $ &c. $+ \dfrac{1}{2^2} \times \dfrac{C^2}{6} = \dfrac{C^2}{6}$;
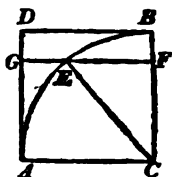
$\therefore \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + $ &c. $= \dfrac{C^2}{6} - \dfrac{C^2}{24} = \dfrac{C^2}{8}$.

In the same manner the sum of the reciprocals of all the even powers of 1, 3, 5, 7, &c. may be found.

Prob. 74.

To compare the momenta of a sphere and its circumscribed cylinder, whilst they revolve round a common axis.

Let $AEBC$ be the quadrant of a circle, whose semi-diameter is CA, or CB. Complete the square $ACBD$; draw any line GEF parallel to AC, cutting AD, BC, in G and F, and the quadrantile arc in E. Join EC. Now let the segments $AEFC$ of the quadrant, and $AGFC$ of the square, revolve about the axis CF, and generate segments of the sphere and circumscribed cylinder. Take $CF=x$, $FE \doteq y$, $CA = r$. Then, the momenta being as the quantities of matter and the velocity jointly, the momentum of the spherical segment $AEFC$: the momentum of the cylindrical segment $AGFC$ :: $f. y^3\dot{x}$: $f. r^2\dot{x}$.

$$:: f. \dot{x} . (r^2 - x^2)^{\frac{3}{2}} : f. r^3\dot{x}$$

$$:: f. r^2\dot{x} . (r^2 - x^2)^{\frac{1}{2}} - f. x^2\dot{x} . (r^2 - x^2)^{\frac{1}{2}} : f. r^3\dot{x}.$$

Now the $f. r^2\dot{x} . (r^2 - x^2)^{\frac{1}{2}} = r^2 \times$ area $ACFE$; and

$$f. x^2\dot{x} . (r^2 - x^2)^{\frac{1}{2}} = \frac{r^2 \times \text{area } ACFE - x . (r^2 - x^2)^{\frac{3}{2}}}{4}$$

(Fluent 24); $\therefore$ the third term in this proportion $=$

$\frac{3}{4}r^2 \times$ area $CFEA + \frac{x . (r^2 - x^2)^{\frac{3}{2}}}{4}$. Let $x=r$; and we

have, the momentum of the sphere : the momentum

of the cylinder :: $\frac{3r^2}{4} \times$ quadrant $AEBC$: r^4; or,

(if $Q =$ the arc of the quadrant) :: $\frac{3r^3}{8} \times Q$: r^4

$$:: 3Q : 8r.$$

SECTION III.

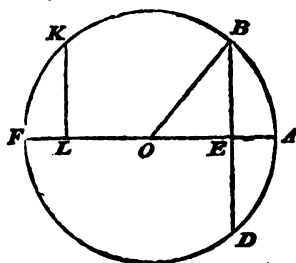### PROB. 75.

To resolve $v^{2n} - 2xv^n + 1 = 0$ into its quadratic divisors, x being equal to or less than unity.

Let AB and AK be two circular arcs, which are to each other as $1 : n$; let $AB = z$, $AK = nz$, $OB = 1$, OE the cosine of $AB = y$, and OL the cosine of $AK = x$. Then

$$\dot{z} = \frac{-\dot{y}}{(1-y^2)^{\frac{1}{2}}} \text{ (Art. 44.), and } n\dot{z} = \frac{-\dot{x}}{(1-x^2)^{\frac{1}{2}}}; \quad \therefore \frac{\dot{x}}{(1-x^2)^{\frac{1}{2}}} = \frac{n\dot{y}}{(1-y^2)^{\frac{1}{2}}};$$

and multiplying each denominator by $(-1)^{\frac{1}{2}}$, $\dfrac{\dot{x}}{(x^2-1)^{\frac{1}{2}}} =$

$\dfrac{n\dot{y}}{(y^2-1)^{\frac{1}{2}}}$; $\therefore$ hyp. log. $x + (x^2 - 1)^{\frac{1}{2}} = n \times$ hyp. log. $y +$

$(y^2 - 1)^{\frac{1}{2}}$ (Art. 43.); hence $x + (x^2 - 1)^{\frac{1}{2}} = \left(y + (y^2 - 1)^{\frac{1}{2}}\right)^n$, by the nature of logarithms.

Now let $v = y + (y^2 - 1)^{\frac{1}{2}}$; $\therefore v - y = (y^2 - 1)^{\frac{1}{2}}$, and $v^2 - 2yv + y^2 = y^2 - 1$; $\therefore v^2 - 2yv + 1 = 0$. Also $v^n = \left(y + (y^2 - 1)^{\frac{1}{2}}\right)^n = x + (x^2 - 1)^{\frac{1}{2}}$; $\therefore v^n - x = (x^2 - 1)^{\frac{1}{2}}$, and $v^{2n} - 2xv^n + 1 = 0$, the given equation; and since v is the same in both equations, one quadratic divisor is $v^2 - 2yv + 1 = 0$.

Now x is not only the cosine of AK, but of $360 + AK$, of $2 \times 360 + AK$, &c.; $\therefore y$ is not only the cosine of

$\dfrac{AK}{n}$, but of $\dfrac{360 + AK}{n}$, of $\dfrac{2 \times 360 + AK}{n}$, &c.   Call these cosines a, b, c, &c.; then $v^{2n} - 2xv^n + 1 = (v^2 - 2av + 1) . (v^2 - 2bv + 1) . (v^2 - 2cv + 1) . \&c.$

There can only be n different values of y; for after taking n arcs, $\dfrac{AK}{n}$, $\dfrac{360 + AK}{n}$, &c., the same cosines will recur.

Cor. 1.   If AK be taken equal to the whole, or to half of the circumference, the equation is $v^{2n} \mp 2v^n + 1 = 0$, and its square root is $v^n \mp 1 = 0$.   But every equation, which is a square, has another root equal to each of its roots; therefore the roots of $v^n \mp 1 = 0$, are found in the same manner.

Cor. 2.   The quadratic divisors of $v^{2n} - 2xr^n v^n + r^{2n} = 0$, are found in the same way; for this equation is merely the equation $v^{2n} - 2xv^n + 1 = 0$, having its roots multiplied by r.   Hence, multiply the roots of the above quadratic divisors by r, and we have $v^2 - 2arv + r^2 = 0$, $v^2 - 2brv + r^2 = 0$, &c. for the required divisors.

Prob. 76.

To demonstrate Cotes' Properties of the Circle.

1.   If any point P be taken in the radius of a circle AO, or the radius produced, and the circumference be divided
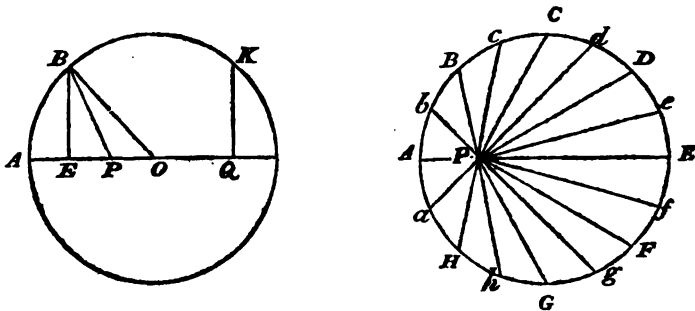
into n equal parts in B, C, D, &c.; then $AO^n - PO^n$, or $PO^n - AO^n = PB$, $PC \times PD \times$ &c.

By Prob. 75, the quadratic divisors of $v^{2n} - 2xv^n + 1 =$ $(v^2 - 2av + 1)$, $(v^2 - 2bv + 1)$, &c. Assume a point P in the radius, and draw PB to the circumference; let $v = PO$, $y = OE$, and the radius $= 1$. Then $BO^2 = OP^2 + PB^2 +$ $2OP \times PE$, or $1^2 = v^2 + BP^2 + 2v \times (y - v)$; $\therefore BP^2 =$ $1^2 - v^2 - 2v \cdot (y - v) = v^2 - 2yv + 1$. Also y is the cosine of $\dfrac{AK}{n}$, $\dfrac{AK + 360}{n}$, $\dfrac{AK + 2 \times 360}{n}$, &c., whose cosines are a, b, c, &c., and $(v^2 - 2av + 1) \cdot (v^2 - 2bv + 1)$. &c. $= v^{2n} - 2xv^n + 1$. Let $AK =$ the whole circumference C; then these arcs are $\dfrac{C}{n}$, $\dfrac{2C}{n}$, $\dfrac{3C}{n}$, &c., or $\dfrac{1^{th}}{n}$, $\dfrac{2^{th}}{n}$, $\dfrac{3^{th}}{n}$, &c. parts of the circumference; that is, if the circumference be divided into n equal parts in B, C, D, &c., the cosines of AB, AC, AD, &c. are a, b, c, &c. and $x = 1$. Hence $PB^2 = v^2 - 2av + 1$, $PC^2 = v^2 - 2bv + 1$, &c.; $\therefore PB^2 \times PC^2 \times PD^2 \times$ &c. $= v^{2n} - 2v^n + 1$, and $PB \times PC \times PD \times$ &c. $= v^n - 1^n$, or $1^n - v^n = PO^n - AO^n$, or $AO^n - PO^n$, according as P is without or within the circle.

<div align="center">Second Property.</div>

If the whole circumference be divided into $2n$ equal parts in b, c, d, &c., then $AO^n + PO^n = Pb \times Pc \times Pd \times$ &c.

By the preceding case, if P be taken, for example, within the circle, $AO^{2n} - PO^{2n} = Pb \times PB \times Pc \times PC \times Pd \times PD \times$ &c.; but $AO^n - PO^n = PB \times PC \times PD \times$ &c.;

$$\therefore \frac{AO^{2n} - PO^{2n}}{AO^n - PO^n} = \frac{Pb \times PB \times Pc \times PC \times Pd \times PD \times \&c.}{PB \times PC \times PD \times \&c.},$$

or $AO^n + PO^n = Pb \times Pc \times Pd \times$ &c.

Cor. 1. If n be an even number, each semi-circle is equally divided; hence, in the equation, $1^n - v^n =$ $(v^2 - 2av + 1)^{\frac{1}{2}} \cdot (v^2 - 2bv + 1)^{\frac{1}{2}} \cdot$ &c. one value of y is $+1$, and another -1; therefore $1^n - v^n = (1 - v) \cdot (v^2 - 2av + 1)^{\frac{1}{2}}$.

$(v^2 - 2bv + 1)^{\frac{1}{2}}$. &c. $(1+v)$. But each PB has a PH in the other semi-circle corresponding to it; therefore $1^x - v^x = (1 - v^2).(1 - 2av + v^2).(1 - 2bv + v^2)$. &c. to $\frac{n}{2}$ terms. This is the method of finding the n roots of 1; two roots are in this case, $+1$ and -1, obtained by solving the quadratic $1 - v^2$, and all the others obtained from the remaining quadratics are impossible.

COR. 2. If n be an odd number, none of the points of section will fall on the extremity of the semi-circle, and therefore no cosine can equal -1; but the cosine of $\frac{n \times C}{n}$, or of 360^0, $= 1$; $\therefore$ here $1^x - v^x = (1 - v)$. $(1 - 2av + v^2).(1 - 2bv + v^2)$. &c. $\frac{n+1}{2}$ terms. In this case, one value of v in the equation $1^x - v^x = 0$, is 1; the other roots are impossible.

COR. 3. By the second property, $1^x + v^x = (1 - 2av + v^2)^{\frac{1}{2}}$. $(1 - 2bv + v^2)^{\frac{1}{2}}$. &c. to n terms. Let n be even; here, as in the former case, to every Pb in one semi-circle there is a correspondent Pa in the other; but no point of section falls on the extremity of the semi-circle; therefore $1^x + v^x = (1 - 2av + v^2).(1 - 2bv + v^2)$, &c. to $\frac{n}{2}$ terms. By solving these quadratic divisors of $1^x + v^x = 0$, we obtain the n roots of -1, and they are all impossible when n is even.

COR. 4. Let n be odd. Since the arcs on each side of A are equal, and no point of section fell upon the extremity of the semi-circle in the last Corollary, one arc will now be a semi-circle, and one value of $v = -1$; hence $1^x + v^n = (1 + v).(1 - 2av + v^2).(1 - 2bv = v^2)$. &c.

$\dfrac{n+1}{2}$ terms. In this case, therefore, one root of the equation $1^n + v^n = 0$ is -1, and the rest are impossible.

PROB. 77.

To resolve $\dfrac{1}{v^{2n} - 2xv^n + 1}$, x being less than 1, into its quadratic divisors.

Let $\dfrac{1}{v^{2n} - 2xv^n + 1} = \dfrac{A}{m-v} + \dfrac{B}{p-v} + \dfrac{C}{q-v} + \&c.$; then

$A \cdot (p-v) \cdot (q-v) \cdot \&c. + B \cdot (m-v) \cdot (q-v) \cdot \&c. + C \cdot (m-v) \cdot (p-v) \cdot \&c. = 1$. Substitute m for v; then $A = $

$\dfrac{1}{(p-m) \cdot (q-m) \cdot \&c.}$. Substitute p for v; then $B = $

$\dfrac{1}{(m-p) \cdot (q-p) \cdot \&c.}$. Now as $v^{2n} - 2xv^n + 1 = (m-v) \cdot$

$(p-v) \cdot \&c.$, take the fluxion on each side, and divide by $\dot{v}$; then $-2nv^{2n-1} + 2nxv^{n-1} = (m-v) \cdot (p-v) \cdot \&c. + (p-v) \cdot (q-v) \cdot \&c.$ Hence, if m be written for v, $-2nm^{2n-1} + 2nxm^{n-1} = (p-m) \cdot (q-m) \cdot \&c.$; but $A = $

$\dfrac{1}{(p-m) \cdot (q-m) \cdot \&c.}$; therefore $A = \dfrac{1}{2nxm^{n-1} - 2nm^{2n-1}} = $

$\dfrac{m}{2nxm^n - 2nm^{2n}}$; and $B = \dfrac{p}{2nxp^n - 2np^{2n}}$, or $A = \dfrac{1}{2n} \times$

$\dfrac{m}{xm^n - m^{2n}}$, and $B = \dfrac{1}{2n} \times \dfrac{p}{xp^n - p^{2n}}$. But by supposition,

$m^{2n} - 2xm^n + 1 = 0$; $\therefore xm^n - m^{2n} = 1 - xm^n$. Hence

$A = \dfrac{1}{2n} \times \dfrac{m}{1 - xm^n}$, and $B = \dfrac{1}{2n} \times \dfrac{p}{1 - xp^n}$. Now $\dfrac{A}{m-v} +$

$\dfrac{B}{p-v} = \dfrac{pA + mB - (A+B) \cdot v}{mp - (m+p) \cdot v + v^2}$; and $1 - 2av + v^2$, a

quadratic divisor of $1-2xv^n+v^{2n}$, is equal to $(m-v)$. $(p-v)$; therefore $mp=1$, and $m+p=2a$; and we have

$$\frac{A}{m-v}+\frac{B}{p-v}=\frac{pA+mB-(A+B).v}{1-2av+v^2}.$$ The object then

is to calculate $pA+mB$ and $A+B$. Now $pA=\frac{1}{2n}$.

$\frac{pm}{1-xm^n}$, or $=\frac{1}{2n}\times\frac{1}{1-xm^n}$; and $mB=\frac{1}{2n}\times\frac{1}{1-xp^n}$;

therefore $pA+mB=\frac{1}{2n}\times \quad : \frac{1}{1-xm^n}+\frac{1}{1-xp^n}=\frac{1}{2n}\times$

$\dfrac{2-x.(m^n+p^n)}{1-x.(m^n+p^n)+x^2m^np^n}$. But $m^{2n}-2xm^n+1=0$;

$\therefore m^n-2x+\dfrac{1}{m^n}=0$; also $p^n=\dfrac{1}{m^n}$; $\therefore m^n+p^n=2x$;

hence $pA+mB=\dfrac{1}{2n}\times\dfrac{2-2x^2}{1-2x^2+x^2}=\dfrac{1}{n}\times\dfrac{1-x^2}{1-x^2}=\dfrac{1}{n}$.

Again, $A=\dfrac{1}{2n}\times\dfrac{m}{1-xm^n}$, and $B=\dfrac{1}{2n}\times\dfrac{p}{1-xp^n}$; $\therefore A+$

$B=\dfrac{1}{2n}\times \quad : \dfrac{m}{1-xm^n}+\dfrac{p}{1-xp^n}=$ (since $mp=1$)$\dfrac{1}{2n}\times \quad :$

$\dfrac{m+p-x.(m^{n-1}+p^{n-1})}{1-x.(m^n+p^n)+x^2.m^np^n}$. Now $m^n+p^n=2x$, where x

is the cosine of an arc, which is to the arc whose cosine is $a :: n : 1$; for the same reason $m^{n-1}+p^{n-1}=2e$, where e is the cosine of an arc, which is to the arc whose cosine is $a :: n-1 : 1$; therefore, since $m+p=2a$,

$A+B=\dfrac{1}{2n}\times\dfrac{2a-2xe}{1-2x^2+x^2}=\dfrac{1}{2n}\times\dfrac{2a-2xe}{1-x^2}$. Hence

$\dfrac{A}{m-v}+\dfrac{B}{p-v}=\dfrac{\dfrac{1}{n}-\dfrac{a-xe}{n-nx^2}\times v}{1-2av+v^2}$; $\therefore \dfrac{1}{v^{2n}-2xv^n+1}=$

$$\frac{\dfrac{1}{n} - \dfrac{a-xe}{n-nx^2} \times v}{1 - 2av + v^2} + \frac{\dfrac{1}{n} - \dfrac{b-xf}{n-nx^2} \times v}{1 - 2bv + v^2} + \&c. \; ; \; \text{where } f \text{ is}$$

found from b, in the same manner that e is obtained from a.

Cor. If x be negative, the quantity to be resolved is

$$\frac{1}{v^{2n} + 2xv^n + 1}; \; \text{it} = \frac{\dfrac{1}{n} - \dfrac{a+xe}{n-nx^2} \times v}{1 - 2av + v^2} + \frac{\dfrac{1}{n} - \dfrac{b+xf}{n-nx^2} \times v}{1 - 2bv + v^2} +$$

&c.

In the same way $\dfrac{1}{1 \pm v^n}$ may be resolved; or it may be

done by the following method;

PROB. 78.

To resolve $\dfrac{1}{1 - v^n}$ into its quadratic divisors, where n is

even.

By Cor. 1. Prob. 76, $1 - v^n = (1 - v^2) . (1 - 2av + v^2)$.

$(1 - 2bv + v^2) . \&c.$ to $\dfrac{n}{2}$ terms; therefore their hyperbolic

logarithms are equal, or hyp. log. $1 - v^n =$ hyp. log. $(1 - v^2)$
$+$ hyp. log. $(1 - 2av + v^2) +$ hyp. log. $(1 - 2bv + v^2) + \&c.$;
therefore the fluxions on each side are equal; or,

$$\frac{-nv^{n-1}\dot{v}}{1-v^n} = \frac{-2v\dot{v}}{1-v^2} + \frac{-2a\dot{v} + 2v\dot{v}}{1 - 2av + v^2} + \frac{-2b\dot{v} + 2v\dot{v}}{1 - 2bv + v^2}, \; \text{to} \; \frac{n}{2}$$

terms.

Divide by $\dfrac{\dot{v}}{v}$, and

$$\frac{-nv^n}{1-v^n} = \frac{-2v^2}{1-v^2} + \frac{-2av + 2v^2}{1 - 2av + v^2} + \frac{-2bv + 2v^2}{1 - 2bv + v^2} \left(\frac{n}{2}\right).$$

Subtract each side from n, viz. the first from n, and each term on the other side from 2, so that the whole may be subtracted from $\dfrac{2 \times n}{2}$, or n; hence,

$$\frac{n}{1-v^n} = \frac{2}{1-v} + \frac{2-2av}{1-2av+v^2} + \frac{2-2bv}{1-2bv+v^2} + \&c. \left(\frac{n}{2}\right).$$

$$\therefore \frac{1}{1-v^n} = \frac{\dfrac{2}{n}}{1-v^2} + \frac{\dfrac{2}{n}-\dfrac{2av}{n}}{1-2bv+v^2} + \frac{\dfrac{2}{n}-\dfrac{2bv}{n}}{1-2bv+v^2} + \&c. \text{ to}$$

$\dfrac{n}{2}$ terms.

Cor. In the same manner $\dfrac{1}{r^n-v^n}$ may be resolved, where n is an even number.

<center>Prob. 79.</center>

To resolve $\dfrac{1}{1-v^n}$ into its quadratic divisors, where n is odd.

By Cor. 2. Prob. 76, in this case $1-v^n = (1-v).$ $(1-2av+v^2) . (1-2bv+v^2) . \&c. \text{ to } \dfrac{n+1}{2}$ terms; $\therefore$ hyp. log. $1-v^n =$ hyp. log. $(1-v) +$ hyp. log. $(1-2av+v^2) +$ hyp. log. $(1-2bv+v^2) + \&c. \left(\dfrac{n+1}{2}\right)$; therefore their fluxions are equal, or

$$\frac{-nv^{n-1}\dot{v}}{1-v^n} = \frac{-\dot{v}}{1-v} + \frac{-2a\dot{v}+2v\dot{v}}{1-2av+v^2} + \frac{-2b\dot{v}+2v\dot{v}}{1-2bv+v^2} + \&c.$$

$\left(\dfrac{n+1}{2}\right).$

Divide each side by $\dfrac{v}{v}$; then,

$$\frac{-nv^n}{1-v^n}=\frac{-.v}{1-v}+\frac{-2av+2v^2}{1-2av+v^2}+\frac{-2bv+2v^2}{1-2bv+v^2}\ \left(\frac{n+1}{2}\right).$$

Subtract each side from n; viz. the first from n, the first term of the other side from 1, and the remaining $\dfrac{n-1}{2}$ terms, each from 2; so that the whole may be subtracted from $1+\dfrac{n-1}{2}\times 2$, or n; and the result is

$$\frac{n}{1-v^n}=\frac{1}{1-v}+\frac{2-2av}{1-2av+v^2}+\frac{2-2bv}{1-2bv+v^2}+\&c.\ \left(\frac{n+1}{2}\right);$$

$$\therefore\frac{1}{1-v^n}=\frac{\dfrac{1}{n}}{1-v}+\frac{\dfrac{2}{n}-\dfrac{2av}{n}}{1-2av+v^2}+\frac{\dfrac{2}{n}-\dfrac{2bv}{n}}{1-2bv+v^2}+\&c.$$

$$\left(\frac{n+1}{2}\right).$$

In the same manner $\dfrac{1}{r^n-v^n}$ may be resolved, where n is an odd number.

PROB. 80.

To resolve $\dfrac{1}{1+v^n}$ into its quadratic divisors, where n is even.

By Cor. 3. Prob. 76, in this case,

$$1^n+v^n=(1-2av+v^2).(1-2bv+v^2).(1-2cv+v^2).\&c.$$

$\dfrac{n}{2}$ terms; then, by the same process as in Prob. 78, we get,

$$\frac{1}{1+v^n}=\frac{\dfrac{2}{n}-\dfrac{2av}{n}}{1-2av+v^2}+\frac{\dfrac{2}{n}-\dfrac{2bv}{n}}{1-2bv+v^2}+\&c.\ \text{to}\ \frac{n}{2}\ \text{terms.}$$

PROB. 81.

To resolve $\dfrac{1}{1+v^n}$ into its quadratic divisors, where n is odd.

By Prob. 76. Cor. 4, in this case,

$$1^n + v^n = (1+v) \cdot (1 - 2av + v^2) \cdot (1 - 2bv + v^2) \cdot \&c. \ \frac{n+1}{2}$$

terms; then, by the same process as in Prob. 79, we get,

$$\frac{1}{1+v^n} = \frac{\dfrac{1}{n}}{1+v} + \frac{\dfrac{2}{n} - \dfrac{2av}{n}}{1 - 2av + v^2} + \frac{\dfrac{2}{n} - \dfrac{2bv}{n}}{1 - 2bv + v^2} + \&c.$$

$\dfrac{n+1}{2}$ terms.

In the same manner $\dfrac{1}{r^n + v^n}$ may be resolved.

PROB. 82.

Given the sine of an arc, to find the sine of n times that arc.

If $AB = z$, $AK = nz$, $OE = y$, and OL the cosine of $AK = x$; then, by Prob. 75, the radius being 1, we have $x + (x^2 - 1)^{\frac{1}{2}} = \left(y + (y^2 - 1)^{\frac{1}{2}} \right)^n$, or $x + (x^2 - 1)^{\frac{1}{2}} = y^n + n y^{n-1} \cdot (y^2 - 1)^{\frac{1}{2}} + n \cdot \dfrac{(n-1)}{2} y^{n-2} \cdot (y^2 - 1) + n \cdot \dfrac{(n-1)}{2} \cdot \dfrac{(n-2)}{3} \cdot$

$y^{n-3} \cdot (y^2 - 1)^{\frac{1}{2}} \cdot (y^2 - 1) + \&c.$ This equation consists of quantities, part of which is possible, and part impossible; hence the possible and impossible parts on each side are respectively equal. Therefore, taking the
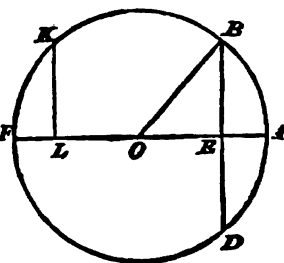
impossible parts, we have $(x^2 - 1)^{\frac{1}{2}} = ny^{n-1} \cdot (y^2 - 1)^{\frac{1}{2}}$

$+ n \cdot \dfrac{(n-1)}{2} \cdot \dfrac{(n-2)}{3} \cdot y^{n-3} \cdot (y^2 - 1)^{\frac{1}{2}} \cdot (y^2 - 1) +$ &c.

Multiply both sides by $(-1)^{\frac{1}{2}}$; then $(1 - x^2)^{\frac{1}{2}} = ny^{n-1} \cdot$

$(1 - y^2)^{\frac{1}{2}} + n \cdot \dfrac{(n-1)}{2} \cdot \dfrac{(n-2)}{3} \cdot y^{n-3} \cdot (1 - y^2)^{\frac{1}{2}} \cdot (y^2 - 1)$

$+$ &c. $= \Big($ if $s = (1 - y^2)^{\frac{1}{2}}$, or $-s^2 = (y^2 - 1)\Big) ny^{n-1}s -$

$n \cdot \dfrac{(n-1)}{2} \cdot \dfrac{(n-2)}{3} y^{n-3} s^3 +$ &c., which is the sine of

AK, n times the arc AB.

PROB. 83.

Given the sine of an arc, to find the sine of an nth part of that arc.

By the same notation, since $x + (x^2 - 1)^{\frac{1}{2}} =$

$\Big(y + (y^2 - 1)^{\frac{1}{2}}\Big)^{n}$; $\therefore y + (y^2 - 1)^{\frac{1}{2}} = \Big(x + (x^2 - 1)^{\frac{1}{2}}\Big)^{\frac{1}{n}} =$

$x^{\frac{1}{n}} + \dfrac{x^{\frac{1-n}{n}} \cdot (x^2 - 1)^{\frac{1}{2}}}{n} + \dfrac{1}{n} \cdot \dfrac{(1-n)}{2n} \cdot x^{\frac{1-2n}{n}} \cdot (x^2 - 1) + \dfrac{1}{n} \cdot$

$\dfrac{(1-n)}{2n} \cdot \dfrac{(1-2n)}{3n} \cdot x^{\frac{1-3n}{n}} (x^2 - 1)^{\frac{1}{2}} \cdot (x^2 - 1) +$ &c. Make

the impossible parts equal; multiply by $(-1)^{\frac{1}{2}}$, and let

$S = (1 - x^2)^{\frac{1}{2}}$, the sine of AK; then $(1 - y^2)^{\frac{1}{2}} = \dfrac{x^{\frac{1-n}{n}} S}{n} -$

$\dfrac{1 \cdot (1-n) \cdot (1-2n)}{n \cdot 2n \cdot 3n} \cdot x^{\frac{1-3n}{n}} S^3 +$ &c., the sine of AB.

PROB. 84.

Given the sine or cosine of an arc, to find the cosine of n times that arc; or, given the cosine of AB, to find that of AK.

Assume the possible parts of the equation in Prob. 82.

equal; then $x = y^n + n \cdot \dfrac{(n-1)}{2} \cdot y^{n-2} \cdot (y^2 - 1) + \&c. =$

$y^n - n \cdot \dfrac{(n-1)}{2} \, y^{n-2} s^2 + \&c.$, the cosine of AK.

PROB. 85.

Given the sine or cosine of an arc, to find the cosine of an nth part of that arc; or, given the cosine of AK, to find that of AB.

Make the possible parts of the equation, in Prob. 83.

equal; then $y = x^{\frac{1}{n}} + \dfrac{1}{n} \cdot \dfrac{(1-n)}{2n} \cdot x^{\frac{1-2n}{n}} (x^2 - 1) + \&c.$, or

$y = x^{\frac{1}{n}} - \dfrac{(1-n)}{n \cdot 2n} \cdot x^{\frac{1-2n}{n}} S^2 + \&c.$, the cosine of AB.

PROB. 86.

Given the sine or cosine, and therefore the tangent of an arc, to find the tangent of n times that arc.

Let $t =$ the tangent of AB; then, by Trigonometry, $t = \dfrac{s}{y}$ to radius 1; therefore the tangent of $AK =$

$\dfrac{S, \; AK}{\text{cos. } AK} =$ (by Probs. 82. and 84.)

$$\dfrac{n y^{n-1} s - n \cdot \dfrac{(n-1)}{2} \cdot \dfrac{(n-2)}{3} y^{n-3} s^3 + \&c.}{y^n - n \cdot \dfrac{(n-1)}{2} \cdot y^{n-2} s^2 + \&c.} =$$

(dividing both the numerator and denominator by y^n)

$$\dfrac{\dfrac{ns}{y} - n \cdot \dfrac{(n-1)}{2} \cdot \dfrac{(n-2)}{3} \cdot \dfrac{s^3}{y^3} + \&c.}{1 - n \cdot \dfrac{(n-1)}{2} \cdot \dfrac{s^2}{y^2} - \&c.} =$$

$$nt - n \cdot \frac{(n-1)}{2} \cdot \frac{(n-2)}{3} \, t^3 + \&c.$$

$$\overline{1 - n \cdot \frac{(n-1)}{2} \cdot t^2 + \&c.}$$

PROB. 87.

Given the same, to find the tangent of an nth part of the arc.

In the same manner as before, by Probs. 83. and 85, the tangent of $AB = \dfrac{\text{sine of } AB}{\text{cos. } AB} =$

$$\frac{\dfrac{x^{\frac{1-n}{n}} S}{n} - \dfrac{1 \cdot (1-n) \cdot (1-2n)}{n \cdot 2n \cdot 3n} \cdot x^{\frac{1-3n}{n}} S^3 + \&c.}{x^{\frac{1}{n}} - \dfrac{(1-n)}{n \cdot 2n} \cdot x^{\frac{1-2n}{n}} S^2 + \&c.} = \text{(dividing}$$

by $x^{\frac{1}{n}}$) $\dfrac{\dfrac{S}{nx} - \dfrac{1 \cdot (1-n) \cdot (1-2n)}{n \cdot 2n \cdot 3n} \cdot \dfrac{S^3}{x^3} + \&c.}{1 - \dfrac{(1-n)}{n \cdot 2n} \cdot \dfrac{S^2}{x^2} + \&c.}$ (if T be the

tangent of AK) $= \dfrac{\dfrac{T}{n} - \dfrac{1 \cdot (1-n) \cdot (1-2n)}{n \cdot 2n \cdot 3n} \cdot T^3 + \&c.}{1 - \dfrac{(1-n)}{n \times 2n} \cdot T^2 + \&c.}$.

COR. And since the secant of an arc $= \dfrac{r^2}{\text{cosine}}$, if the sine or cosine, and therefore the secant, of an arc be given, we can find the secant of n times that arc, or of an nth part of it.

SECTION IV.

PROB. 88.

Required the quantity of matter in a sphere, whose density varies as the nth power of the distance from the center.

Let $d =$ the density of the sphere at the surface, $r =$ the radius, $x =$ any variable distance from the center, $p =$ the area of a circle, whose radius is 1. Then the surface of a sphere, whose radius is $x, = 4px^2$; and since $r^n : x^n :: d :$ density at the distance x, this density $= \dfrac{dx^n}{r^n}$; therefore the fluxion of that quantity of matter $= 4px^2\dot{x} \times \dfrac{dx^n}{r^n}$, and the content $= \dfrac{4pdx^{n+3}}{(n+3) . r^n}$; or the quantity of matter in the whole sphere $= \dfrac{4pdr^3}{n+3}$.

COR. 1. The quantities of matter in two spheres, whose densities are equal at the surface, and vary according to the same law of the distance from the center, are as the cubes of the radii.

COR. 2. If $n = 0$, or the density is constant, the content $= \dfrac{4pdr^3}{3}$.

COR. 3. If two equal spheres have the same density at the surface, and the density of one be constant, whilst that of the other varies as the distance from the center,

The content of the former : that of the latter $:: \dfrac{4pdr^3}{3} : \dfrac{4pdr^3}{4}$

$$:: \quad 4 \quad : \quad 3.$$

Cor. 4. If the density of a globe vary as the nth power of the distance from the center, and the density of its circumscribing cylinder be uniformly the same with that of the surface of the globe, The content of the globe :

content of the cylinder :: $\dfrac{4pdr^3}{n+3}$: $2pdr^3$

$$:: \quad 2 \quad : n+3.$$

Cor. 5. If the quantity of matter in a sphere be equal to that of its circumscribing cylinder, the density of the cylinder being uniformly the same with that of the surface of the sphere, the law of the density may be determined.

For $\dfrac{4pdr^3}{n+3} = 2pdr^3$; $\therefore 2 = n+3$, and $n = -1$, or the density varies inversely as the distance from the center.

Prob. 89.

If a body descend by gravity down the quadrant of a circle AC, the radius AB being parallel to the horizon; to find where the velocity in the direction BC is a maximum.

Let $ML = y$, $BL = r$, $AL = z$; then the velocity at $L \propto ML^{\frac{1}{2}} \propto y^{\frac{1}{2}}$; and the velocity in the curve : velocity in the direction BC :: LT : vT, (TN being drawn parallel and near to ML, and Lv perpendicular,) or :: LB : BM :: r : $(r^2 - y^2)^{\frac{1}{2}}$; $\therefore$ the velocity in the

direction $BC \propto \dfrac{(r^2 y - y^3)^{\frac{1}{2}}}{r}$, a maximum; hence $r^2 y - y^3$

is a maximum; its fluxion $= 0$, and $y = \dfrac{r}{3^{\frac{1}{2}}} = r \times \dfrac{1}{3^{\frac{1}{2}}}$.

PROB. 90.

To find the same, if a body descend down the arc of a cycloid.

The same assumption being made, velocity in the direction LT : velocity in the direction BC :: LT : vT; by similar triangles, :: RC : CS :: BC : CR :: $BC^{\frac{1}{2}}$: $CS^{\frac{1}{2}}$; that is, :: $a^{\frac{1}{2}}$: $(a-y)^{\frac{1}{2}}$; hence the velocity in the
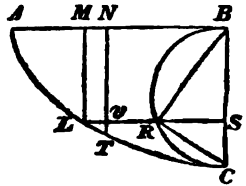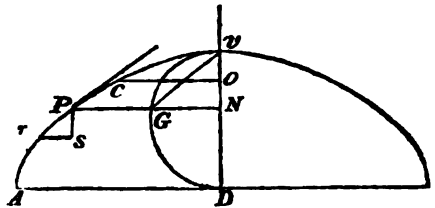
direction BC varies as $\dfrac{(ay-y^2)^{\frac{1}{2}}}{a^{\frac{1}{2}}}$, a maximum; therefore

the fluxion of $ay - y^2 = 0$, or $2y\dot{y} = a\dot{y}$, and $y = \dfrac{a}{2}$.

PROB. 91.

To find where a body, moving on the convex side of a cycloid VPA, will fly off, its motion being supposed to begin from the point C.

After leaving the cycloid, the body will describe a parabola, and its horizontal velocity will therefore be constant; hence the fluxion of the velocity in a horizontal direction, at the point where the body leaves the curve, must $=0$. Suppose that point to be P; let $VD=a$, $VO=d$, $VN=x$; then $VG = (ax)^{\frac{1}{2}}$, and $GN= (ax-x^2)^{\frac{1}{2}}$. Now the velocity at $P =$ that acquired in falling through ON, or $x - d$, $= \left(4m \cdot (x - d)\right)^{\frac{1}{2}}$,

where $m = 16\frac{1}{12}$ feet. And this velocity $\left(4m \cdot (x - d)\right)^{\frac{1}{2}}$:

horizontal velocity :: Pr : sr :: VG : GN

$$:: (ax)^{\frac{1}{2}} : (ax - x^2)^{\frac{1}{2}};$$

∴ the horizontal velocity $= \dfrac{\left(4m \cdot (x - d)\right)^{\frac{1}{2}} \cdot (ax - x^2)^{\frac{1}{2}}}{(ax)^{\frac{1}{2}}}$,

or $\propto (x - d)^{\frac{1}{2}} \cdot (a - x)^{\frac{1}{2}}$, whose fluxion $= 0$; ∴ the fluxion of $(x - d) \cdot (a - x) \doteq 0$, or $\dot{x} \cdot (a - x) - \dot{x} \cdot (x - d) = 0$;

∴ $a - x = x - d$, and $x = \dfrac{a + d}{2}$.

Cor. If the body be just put in motion from V, $d = 0$, and $x = \dfrac{a}{2}$; or the body will leave the cycloid at a point determined by drawing the ordinate NP from the center of the generating circle.

Prob. 92.

A body begins to roll from A on the quadrant of a circle with the velocity acquired by falling through LA; to determine the point where it will fly off from the curve.

Let $LA = d$, $AE = x$, $AC = a$; then the velocity in the curve at P, as before, $=$ $\left(4m \cdot (d + x)\right)^{\frac{1}{2}}$; and $\left(4m \cdot (d + x)\right)^{\frac{1}{2}}$: horizontal velocity :: Ps : rs :: PC : CE,
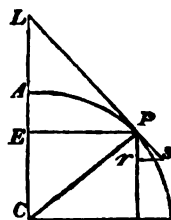
$$:: a : a - x;$$

∴ the horizontal velocity $\propto (d + x)^{\frac{1}{2}} \cdot (a - x)$, whose fluxion, when the body leaves the curve, $= 0$; and x will be found $= \dfrac{a - 2d}{3}$.

Cor. 1. If $AL = \dfrac{AC}{2}$, $AE = 0$, and the body moves in the direction of a tangent at A. If AL be greater than $\frac{1}{2} AC$, x is negative or the Problem impossible.

Cor. 2. If $AL = 0$, $AE = \dfrac{AC}{3}$; but in this case the body must be supposed to be just set in motion at A, or it would have no tendency to roll on the arc, a tangent at A being parallel to the horizon.

Prob. 93.

Required that point in a parabola, where the linear velocity increases or decreases the fastest.

Let AP be the parabola, S the focus, PY a tangent at P; join SP, and draw SY per-pendicular to the tangent. Now the increment of the velocity varies as the force $\times$ the increment of the time; therefore, *dato tempore*, the increment of velocity varies as the force, and is a maximum, when the tangential force is the greatest.
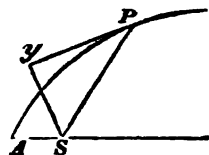
But the force in the direction PS : force in the direction PY :: SP : PY; therefore, since the force in the direction $PS \propto \dfrac{1}{SP^2}$, the tangential force in the direction $PY \propto \dfrac{PY}{SP^3}$, which in this case is a maximum. Let $SA = a$, $SP = x$; then $PY = (x^2 - ax)^{\frac{1}{2}}$; $\therefore \dfrac{(x^2 - ax)^{\frac{1}{2}}}{x^3}$, or $\dfrac{x^2 - ax}{x^6}$, or $\dfrac{x - a}{x^5}$, is a maximum; hence $x^5 \dot{x} - 5x^3 \dot{x} + 5ax^4 \dot{x} = 0$; $\therefore x = \dfrac{5a}{4}$.

Prob. 94.

Required that point in a parabola, where the angular velocity increases or decreases the fastest.

Draw SQ indefinitely near SP, and QT perpendicular to it; the angular velocity varies as $\dfrac{QT}{SP}$; therefore the angular velocity itself is a maximum, when $\dfrac{QT}{SP}$ is a maximum. But the increment of the angular velocity varies as the $\dfrac{\text{increment of } QT}{SP}$, and the increment of QT varies as the force in the direction QT; therefore the increment of angular velocity is a maximum, when $\dfrac{\text{force in direction } QT}{SP}$ is a maximum.
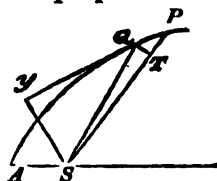
Now force in direction SP : f. in direct. $PY :: SP : PY$, and f. in direc. PY : f. in direct. $QT :: PQ : QT$ or $:: SP : SY$;

$\therefore$ f. in direc. SP : f. in direct. $QT :: SP^{2} : SY \times PY$;

$\therefore$ force in the direction $QT \propto \dfrac{SY \times PY}{SP^{4}}$; and $\dfrac{\text{this force}}{SP} \propto$

$\dfrac{SY \times PY}{SP^{5}}$, which is a maximum.

Let $SA = a$, $SP = x$; then $SY = (ax)^{\frac{1}{2}}$, and $PY = (x^{2} - ax)^{\frac{1}{2}}$; $\therefore \dfrac{(ax)^{\frac{1}{2}} \cdot (x^{2} - ax)^{\frac{1}{2}}}{x^{5}}$, or $\dfrac{(x - a)^{\frac{1}{2}}}{x^{4}}$, or $\dfrac{x - a}{x^{8}}$
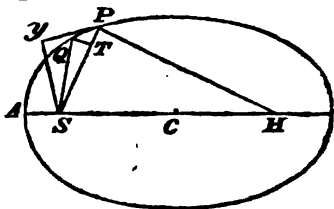
is a maximum; hence $x^{8}\dot{x} - 8x^{8}\dot{x} + 8ax^{7}\dot{x} = 0$, and $x = \dfrac{8a}{7}$.

The same principle might be adopted in the two following Propositions; but it is more convenient to use a different method.

PROB. 95.

Required that point in an ellipse, where the linear velocity increases or decreases the fastest.

Take P some point in the ellipse, whose foci are S, H; join SP, HP, and draw SY perpendicular to a tangent at P. Let $SP=x$, $HP=v$, the semi-major axis $=a$, and semi-minor $=b$. Now the linear velocity $\propto \dfrac{1}{SY} \propto \left(\dfrac{v}{x}\right)^{\frac{1}{2}}$,

from the nature of the ellipse ; $\therefore$ the fluxion of the linear velocity $\propto \dfrac{\dot{v}\,x - \dot{x}\,v}{x\,(v\,x)^{\frac{1}{2}}}$; but $x + v = 2\,a$, or $\dot{v} = -\,\dot{x}$;

$\therefore$ the fluxion of the linear velocity $\propto \dfrac{-\dot{x}.(x+v)}{x\,(v\,x)^{\frac{1}{2}}} \propto \dfrac{\dot{x}}{x\,(v\,x)^{\frac{1}{2}}}$.

But the area SPQ described, *dato tempore*, round S is constant in the same curve (Newton, Prop. 1.); therefore $SP \times QT$ is equal to a constant quantity ; and
$$PT : QT :: PY : SY, \text{ or}$$
$$\dot{x} : QT :: \left(\frac{x^2 v - b^2 x}{v}\right)^{\frac{1}{2}} : b \times \left(\frac{x}{v}\right)^{\frac{1}{2}}, \text{ by the nature of}$$

the ellipse ; $\therefore QT = \dfrac{b\,\dot{x}}{(vx-b^2)^{\frac{1}{2}}}$; $\therefore SP \times QT \propto \dfrac{x\dot{x}}{(vx-b^2)^{\frac{1}{2}}}$,

which is constant, or $\propto 1$; and $\dot{x} \propto \dfrac{(vx-b^2)^{\frac{1}{2}}}{x}$; hence the

expression $\dfrac{\dot{x}}{x\,(vx)^{\frac{1}{2}}} \propto \dfrac{(vx-b^2)^{\frac{1}{2}}}{x^2 \times v^{\frac{1}{2}}x^{\frac{1}{2}}}$, which is a maximum ;

that is, $\dfrac{vx - b^2}{v\,x^5}$ is a maximum, or $\dfrac{1}{x^4} - \dfrac{b^2}{v\,x^5}$; therefore,

taking the fluxion, $\dfrac{-4x^3\dot{x}}{x^9} + \dfrac{b^2.(5x^4\dot{x}v + x^5\dot{v})}{v^2x^{10}} = 0$;

therefore $\dfrac{-4v^2x^5\dot{x}+5b^2x^4v\dot{x}+b^2x^5\dot{v}}{v^2x^{10}}=0$; omit the deno-

minator, and for $\dot{v}$ write $-\dot{x}$; then $5b^2x^4v - b^2x^5 - 4v^2x^5=0$; that is, $5b^2v-b^2x-4v^2x=0$. For x write $2a-v$;

then $5b^2v+b^2v-2ab^2+4v^3-8av^2=0$,

or $4v^3+6b^2v-8av^2-2ab^2=0$,

or $2v^3+4av^2+3b^2v-ab^2=0$,

a cubic equation, in which there will be only one possible value of v.

PROB. 96.

Required that point in an ellipse, where the angular velocity increases or decreases the fastest. (See preceding Fig.)

Draw SQ indefinitely near, and QT perpendicular to SP.

The angular velocity $\propto \dfrac{\text{the area described, }\textit{dato tempore,}}{(\text{dist.})^2}$

$\propto \dfrac{1}{SP^2}$, since the area, *dato tempore*, is constant, $\propto \dfrac{1}{x^2}$;

hence the fluxion of the angular velocity $\propto \dfrac{x\dot{x}}{x^4} \propto \dfrac{\dot{x}}{x^3}$. But

$\dot{x} \propto \dfrac{(vx-b^2)^{\frac{1}{2}}}{x}$; $\therefore \dfrac{\dot{x}}{x^3} \propto \dfrac{(vx-b^2)^{\frac{1}{2}}}{x^4}$, which is a max-

imum, and its fluxion $=0$; that is, $\dfrac{(v\dot{x}+x\dot{v}).x^4}{2.(vx-b^2)^{\frac{1}{2}}} - 4x^3\dot{x}$.

$(vx-b^2)^{\frac{1}{2}}=0$. But $\dot{x}+\dot{v}=0$, and $\dot{x}=-\dot{v}$; $\therefore \dfrac{(v-x).\dot{x}\times x^4}{2(vx-b^2)^{\frac{1}{2}}}=$

$4x^3\dot{x}.(vx-b^2)^{\frac{1}{2}}$; hence $-x^2=7vx-8b^2$. Now for v

write $2a - x$, and x will be found $= \dfrac{7a - (49\,a^2 - 48b^2)}{6}$.

The other value of x is of no use; for it is a quantity greater than x ever can be in the ellipse.

<p align="center">PROB. 97.</p>

To find the point, where the paracentric velocity of a body revolving in any curve is a maximum.

Let PR be the curve, S the center of force, PY a tangent at P, and SY perpendicular to it. Take PQ a small arc in the curve, and QT perpendicular to SP. Then the velocity in the curve : paracentric velocity
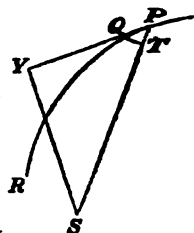
$$:: \quad QP \; : \; PT$$
$$:: \quad SP \; : \; PY;$$

$\therefore$ the paracentric vel. $= \dfrac{\text{vel. in curve} \times PY}{SP}$

$\propto \dfrac{PY}{SY \times SP}$ $\left(\text{for the velocity in the curve} \propto \dfrac{1}{SY}\right) \propto$

$\dfrac{(SP^2 - SY^2)^{\frac{1}{2}}}{SP \times SY}$. Hence, in the case required, this expression, or its square $\dfrac{SP^2 - SY^2}{SP^2 \times SY^2}$, is a maximum.

<p align="center">EXAMPLES.</p>

Ex. 1. Let the curve be a parabola.

Here if $SP = x$, $SY^2 = ax$, and $\dfrac{x^2 - ax}{ax \times x^2}$, or $\dfrac{x - a}{x^2}$, is a maximum; $\therefore x^2 \dot{x} - 2x\dot{x} \cdot (x - a) = 0$; $\therefore x = 2x - 2a$, and $x = 2a = \frac{1}{2}$ the *latus rectum*.

Ex. 2. Let the curve be an ellipse. (Fig. Prob. 95.)

If $SP = x$, $HP = v$, $AC = a$, and the semi-minor axis $= b$;

then $SY^2 = \dfrac{b^2 x}{v}$, and $SP^2 - SY^2 = x^2 - \dfrac{b^2 x}{v} = \dfrac{vx^2 - b^2 x}{v}$,

and $SP^2 \times SY^2 = \dfrac{b^2 x^3}{v}$; therefore $\dfrac{SP^2 - SY^2}{SP^2 \times SY^2} = \dfrac{vx^2 - b^2 x}{b^2 x^3} \propto$

$\therefore \dfrac{vx - b^2}{x^2} \propto \dfrac{(2a - x) \cdot x - b^2}{x^2} \propto \dfrac{2ax - x^2 - b^2}{x^2} \propto \dfrac{2ax - b^2}{x^2} - 1,$

which is a maximum; and $2ax^2 \dot{x} - 2x\dot{x}.(2ax - b^2) = 0$, or

$ax = 2ax - b^2$; $\therefore x = \dfrac{b^2}{a} = \frac{1}{2}$ the *latus rectum*.

PROB. 98.

To compare the angular velocity of the radius vector, with the angular velocity of the perpendicular let fall upon the tangent in any curve.

Let PA be any curve described by a body revolving round a center of force S, PY a tangent at P, and SY perpendicular to it from S. Let SQ be taken indefinitely near to SP, and let PR, PO be respectively the diameter and radius of the circle of curvature at P, and PV the chord passing through S. Then since SY and OP are always parallel, their angular velocities are equal; and
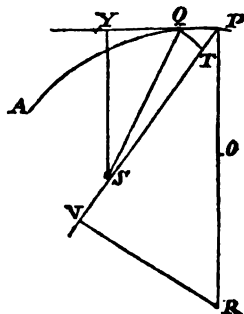
the angular vel. of SP : angular vel. of SY :: $\dfrac{QT}{SP}$: $\dfrac{PQ}{PO}$;

but by similar triangles,

$$QT : QP :: SY : SP$$
$$:: PV : PR; \quad \therefore \text{ the angular}$$

vel. of SP : angular vel. of SY $::\dfrac{PV}{SP}:\dfrac{2PO}{PO}$

$$::PV:2SP$$

$$::\dfrac{2p\dot{x}}{\dot{p}}:2x\quad(\text{Art. 118.})$$

$$::\dfrac{\dot{x}}{x}:\dfrac{\dot{p}}{p}.$$

$::$ (vel.)^2 in the curve : (vel.)^2 in the circle at the same distance. (Art. 162.)

Cor. So long as the angular velocity of the perpendicular is greater than that of the distance, the angle at the center S perpetually increases, and therefore the angle between the tangent and the radius vector diminishes; the angle at P is therefore the least, when the angular velocity of the perpendicular and the distance are equal, or when the velocity in the curve equals that in a circle at the same distance. In the ellipse, for example, the angle contained between the tangent and a line drawn from the focus is a minimum at the extremity of the minor axis.

Prob. 99.

To determine the nature of the curve, down which a body must descend after its fall through a given space by the action of a constant force, so that it may, in the direction of its first descent, describe equal spaces in equal times.

Let the space DA through which the body falls $= a$, and suppose AFC to be the curve required; take FG a small arc, and on DA produced draw FN, GL, CB perpendiculars, and let FH be parallel to AB. Take $AN=x$, $AF=z$, $FN=y$; then $FG=\dot{z}$, $FH=\dot{x}$, $HG=\dot{y}$. Now the velocity in the curve at $F =$ that through DN, and therefore varies as
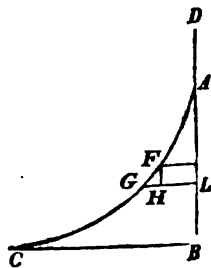
$(a + x)^{\frac{1}{2}}$; and the velocity in the direction FH is constant by supposition, and equal to that acquired down DA ;

$\therefore \dot{z} : \dot{x} :: (a+x)^{\frac{1}{2}} : a^{\frac{1}{2}}$, or $\dot{z}^2 : \dot{x}^2 :: a+x : a$;

that is, $\dot{x}^2 + \dot{y}^2 : \dot{x}^2 :: a+x : a$, and $\dot{y}^2 : \dot{x}^2 :: x : a$;

$\therefore \dot{y} : \dot{x} :: x^{\frac{1}{2}} : a^{\frac{1}{2}}$, and $a^{\frac{1}{2}}\dot{y} = x^{\frac{1}{2}}\dot{x}$; $\therefore a^{\frac{1}{2}}y = \dfrac{2x^{\frac{3}{2}}}{3}$;

or $\dfrac{9ay^2}{4} = x^3$, and the curve is the semi-cubical parabola.

Cor. The *latus rectum* of this semi-cubical parabola $= \dfrac{9a}{4}$.

Prob. 100.

A body describes a curve, whose equation is $a^{n-1}y = x^n$, by a force acting in parallel lines, and in the direction of the ordinates ; required the variation of the velocity.

By Art. 161. $v^2 \propto F \times PV \propto -\ddot{y} \times \dfrac{\dot{z}^2}{-\dot{y}}$ (Art. 103.) $\propto \dot{z}^2$.

Now $\dot{y} = \dfrac{nx^{n-1}\dot{x}}{a^{n-1}}$; $\therefore \dot{y}^2 = \dfrac{n^2}{a^{2n-2}} \times x^{2n-2}\dot{x}^2$, and $\dot{z}^2 = \dot{x}^2 + \dot{y}^2 = \dfrac{(n^2 x^{2n-2} + a^{2n-2}).\dot{x}^2}{a^{2n-2}}$; $\therefore v \propto (n^2 x^{2n-2} + a^{2n-2})^{\frac{1}{2}}$.
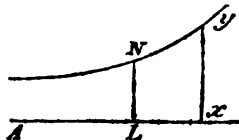
· Prob. 101.

To determine the curve, in which a body revolving by a force, which acts in lines perpendicular to the axis, shall approach to or leave the axis with a velocity always proportional to the ordinates.

Let NY be the curve, Ax the axis, LN and xy ordinates ; let $AL = x$, $LN = y$. Then $\dot{x}$ is constant, and by the Problem $\dot{y} \propto y$; $\therefore \dfrac{\dot{y}}{y}$ is constant, or $\dfrac{\dot{y}}{y} \propto \dot{x}$, a property of the logarithmic curve. (Art. 132.)
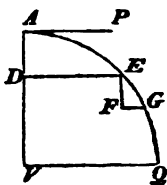
PROB. 102.

A body is projected with a given velocity from A, in a direction AP parallel to the plane VQ; to find the curve described by the body, the attractive force of each particle being supposed constant.

Let AQ be the curve described; assume AV perpendicular to the plane $=a$, VD a variable distance $=x$, DE an ordinate $=y$, $EF = -\dot{x}$, $FG = \dot{y}$, and let the uniform velocity in the direction $AP = b$. Then, if v represent the velocity E in the direction AV, $v\dot{v} \propto -F\dot{x}$; $\therefore v^2 \propto -x + \text{corr.}$, and $v^2 \propto a - x$; $\therefore v \propto (a-x)^{\frac{1}{2}} = d^{\frac{1}{2}} \cdot (a-x)^{\frac{1}{2}}$. But the velocities are as the spaces described uniformly, *dato tempore*; $\therefore -\dot{x} : \dot{y} :: d^{\frac{1}{2}} \cdot (a-x)^{\frac{1}{2}} : b$; therefore $\dot{y} = -\dfrac{b\dot{x}}{d^{\frac{1}{2}} \cdot (a-x)^{\frac{1}{2}}}$, and $y = \dfrac{2b}{d} \cdot (a-x)^{\frac{1}{2}}$; that is, the curve AQ is a parabola, whose vertex is A, and principal *latus rectum* $= \dfrac{4b^2}{d}$.

Cor. This corresponds with the motion of a body projected at the earth's surface, where $d = 4m$, and the *latus rectum* $= \dfrac{b^2}{m}$, m being $16\frac{1}{12}$ feet.

PROB. 103.

Conversely, a body projected in a direction AP parallel to the plane VQ, whose particles attract it according to a certain law, describes a parabola AQ; required the law of the force.

Let $p =$ the parameter of the parabola; then $p \cdot (a-x)$

$=y^2$; $\therefore y=\left(p.(a-x)\right)^{\frac{1}{2}}$, and $\dot{y}=\dfrac{-\frac{1}{2}p\dot{x}}{\left(p.(a-x)\right)^{\frac{1}{2}}}$; $\therefore \dot{y}$:

$-\dot{x} :: \frac{1}{2}p : \left(p.(a-x)\right)^{\frac{1}{2}}$. But $\dot{y} : -\dot{x} ::$ the uniform velocity (b) in the direction AP : the velocity (V) at E ;

$$\therefore b : v :: \tfrac{1}{2}p : \left(p.(a-x)\right)^{\frac{1}{2}};$$

hence $v \propto (a-x)^{\frac{1}{2}}$, and $v^2 \propto a - x$; $\therefore v\dot{v} \propto - \dot{x}$. But $v\dot{v} \propto - F\dot{x}$, where F is the accelerating force ; $\therefore - F\dot{x} \propto -\dot{x}$, and $F \propto 1$, or is constant.

<h2 style="text-align:PROB. 104.</h2>

A body is projected with a given velocity from A in a direction AP parallel to the plane VQ; to find the curve described by the body, the attractive force of each particle varying inversely as the dist.[3] (See preceding Fig.)

Let AQ be the curve described, and assume as before ; then, if v represent the velocity at any distance x from the plane in the direction AV, $v\dot{v} \propto - F\dot{x} \propto \dfrac{-\dot{x}}{x^3}$; $\therefore v^2 \propto$

$\dfrac{1}{x^2}-\dfrac{1}{a^2} \propto \dfrac{a^2 - x^2}{a^2 x^2}$, and $v \propto \dfrac{(a^2-x^2)^{\frac{1}{2}}}{x}$. But the velocity $\propto$ space uniformly described, *dato tempore*; $\therefore - \dot{x} : \dot{y} :: d$. $\dfrac{(a^2 - x^2)^{\frac{1}{2}}}{x} : b$; the factor d being assumed in the third term to make that term of the same dimensions with the fourth ; hence $\dot{y} = \dfrac{bx\dot{x}}{d.(a^2 - x^2)^{\frac{1}{2}}}$, and $y = \dfrac{b}{d}. (a^2 - x^2)^{\frac{1}{2}}$, which is the property of an ellipse; therefore the curve AQ is an ellipse, whose semi-major is $AV = a$, and semi-minor axis $VQ = \dfrac{ab}{d}$.

F F 2

COR. 1. If $b = d$, the curve is a circle.

COR. 2. If the force be repulsive, it may be shewn in the same manner, that the curve is an hyperbola, whose center is V, and vertex A; for in this case $y = \dfrac{b}{d} \cdot (x^2 - a^2)^{\frac{1}{2}}$.

PROB. 105.

Conversely, a body projected in a direction AP parallel to the plane VQ, whose particles attract it according to a certain law, describes an ellipse AQ; required the law of the force.

Here if $c =$ the semi-minor, and a the semi-major axis of the ellipse, $y = \dfrac{c}{a} \cdot (a^2 - x^2)^{\frac{1}{2}}$; $\therefore \dot{y} = \dfrac{c}{a} \cdot \dfrac{-x\dot{x}}{(a^2 - x^2)^{\frac{1}{2}}}$; $\therefore \dot{y} : -\dot{x} :: cx : a \cdot (a^2 - x^2)^{\frac{1}{2}}$. But $\dot{y} : -\dot{x} ::$ the uniform velocity in the direction AP (b) : the velocity towards the plane at E (v); $\therefore b : v :: cx : a \cdot (a^2 - x^2)^{\frac{1}{2}}$; $\therefore v \propto \dfrac{(a^2 - x^2)^{\frac{1}{2}}}{x}$, and $v^2 \propto \dfrac{a^2 - x^2}{x^2}$. Hence $v\dot{v} \propto -\dfrac{2a^2 x \dot{x}}{x^4} \propto -\dfrac{\dot{x}}{x^3}$; but if F represent the accelerating force, $v\dot{v} \propto -F\dot{x}$; $\therefore -F\dot{x} \propto -\dfrac{\dot{x}}{x^3}$, and $F \propto \dfrac{1}{x^3}$.

PROB. 106.

To find the time of vibration of a pendulum in the arc of a circle.

Let D be the point from which the pendulum CD begins its vibrations, and from any point F in the arc DA draw FG parallel to the horizon, meeting the
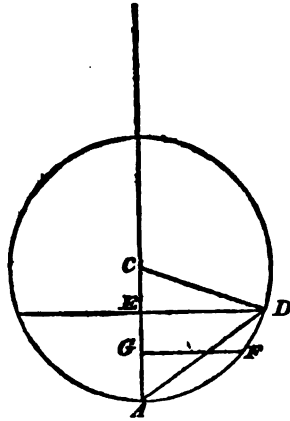
vertical line CA in G. Let $CA=a$, $AG=x$, $AE=b$,

and $AF=z$. Then $\dot{z}=\dfrac{a\dot{x}}{(2ax-x^2)^{\frac{1}{2}}}$;

and since the velocity at $F=$

$\left(4m\,.\,(b-x)\right)^{\frac{1}{2}}$, we have $\dot{T}=-\dfrac{\dot{z}}{V}$

$=\dfrac{-a\dot{x}}{\left(4m\,.\,(b-x)\right)^{\frac{1}{2}}.\,(2ax-x^2)^{\frac{1}{2}}}=$

$\dfrac{-1}{(4m)^{\frac{1}{2}}}\,.\,\dfrac{a\dot{x}}{(bx-x^2)^{\frac{1}{2}}.\,(2a-x)^{\frac{1}{2}}}$,

whose fluent, or T, found by ex-

panding $\dfrac{1}{(2a-x)^{\frac{1}{2}}}$, or $\left(1-\dfrac{x}{2a}\right)^{-\frac{1}{2}}$

by the binomial theorem, $=+\dfrac{pa^{\frac{1}{2}}}{(8m)^{\frac{1}{2}}}\times:1+\dfrac{b}{2\,.\,2\,.\,2a}+$

$\dfrac{3\,.\,3b^2}{2\,.\,2\,.\,4\,.\,4\,.\,(2a)^2}+$ &c., where $p=$ the circumference of

a circle to the diameter 1; therefore the whole time of

vibration, which is double of this, $=p\,.\,\left(\dfrac{a}{2m}\right)^{\frac{1}{2}}\times:1+$

$\dfrac{b}{2\,.\,2\,.\,2a}+\dfrac{3\,.\,3b^2}{2\,.\,2\,.\,4\,.\,4\,.\,(2a)^2}+$ &c.

Cor. The time of descent down the diameter, or the

chord $AD,=\left(\dfrac{2a}{m}\right)^{\frac{1}{2}}$; hence, if the arc in which the pen-

dulum vibrates be very small, or b be supposed evanescent,

we have, time down arc, or $\frac{1}{2}$ time of vibration : time down

the chord $::\dfrac{p}{2}\,.\,\left(\dfrac{a}{2m}\right)^{\frac{1}{2}}:\left(\dfrac{2a}{m}\right)^{\frac{1}{2}}::p:4::$ the cir-

cumference of a circle : four times its diameter.

PROB. 107.

To find the time in which a body will fall by gravity down the arc of a semi-cycloid.

Let $AB = a$, $AC = x$, $AE = z$; then $AD = (ax)^{\frac{1}{2}}$, and $AE = 2(ax)^{\frac{1}{2}}$; therefore the fluxion of the arc $= a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}$. Also $BC = a - x$; ∴ if $m = 16\frac{1}{12}$ feet, velocity
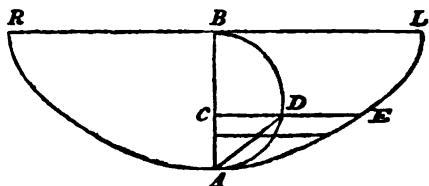
at $E = \left(4m \cdot (a-x)\right)^{\frac{1}{2}}$; $\dot{T} = \dfrac{-\dot{S}}{V} = \dfrac{-a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}}{\left(4m.(a-x)\right)^{\frac{1}{2}}} =$

$\dfrac{-2}{(4ma)^{\frac{1}{2}}} \times \dfrac{\frac{1}{2}a\dot{x}}{(ax-x^2)^{\frac{1}{2}}}$; ∴ (Art. 44.) $T = \dfrac{-2}{(4ma)^{\frac{1}{2}}} \times$

a circular arc of radius $\frac{1}{2}a$, and versed sine x + corr. when $x = a$, $T = 0$; ∴ $T = \dfrac{2}{(4ma)^{\frac{1}{2}}} \times$ a circular arc of

radius $\frac{1}{2}a$, and versed sine a; the remainder vanishing,

when $x = 0$, $= \dfrac{2}{(4ma)^{\frac{1}{2}}} \times \frac{1}{4}$ circumference $=$ time from

L to A; therefore the time from L to $R = \dfrac{4}{(4ma)^{\frac{1}{2}}} \times$

$\frac{1}{4}$ circumference $= \dfrac{2}{(ma)^{\frac{1}{2}}} \times \dfrac{pa}{2}$, if $p = 3.14159$, &c. $=$

$p \times \left(\dfrac{a}{m}\right)$. Now the time down the axis $= \left(\dfrac{a}{m}\right)^{\frac{1}{2}}$; ∴
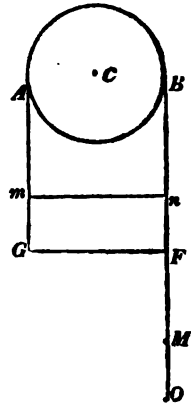
time of vibration : time down axis :: $p \cdot \left(\dfrac{a}{m}\right)^{\frac{1}{2}} : \left(\dfrac{a}{m}\right)^{\frac{1}{2}}$ ::

$p : 1$ :: circumference of a circle : its diameter.

PROB. 108.

If a chain of uniform density be hung over a fixed pulley, and the parts of the chain be unequal, the heavier will descend; given the length of the chain, and the parts of it on each side at the commencement of the motion, to find the time in which it will leave the pulley.

Let AG, BM be the parts of the chain on each side of the pulley at the beginning of the motion. Draw GF parallel to the horizon; then the moving force with which M begins to descend is the part FM. Let M descend to O, and take $Gm = MO$; the moving force then $= nO = FM + 2MO$. Let $L =$ the whole length of the chain, and the force of gravity upon it $= 1$; then, if $FM = 2a$, and $MO = x$, $no = 2a + 2x$, and the accelerating force $= \dfrac{2a + 2x}{L}$. Let $z =$ the space through which a body would fall by gravity to acquire the velocity at O; then $\dot{z} = \dfrac{2a\dot{x} + 2x\dot{x}}{L}$, and $z = \dfrac{2ax + x^2}{L}$; $\therefore$ velocity at $O = \left(\dfrac{4m}{L}\right)^{\frac{1}{2}} \cdot (2ax + x^2)^{\frac{1}{2}}$.

Hence $\dot{T} = \left(\dfrac{L}{4m}\right)^{\frac{1}{2}} \cdot \dfrac{\dot{x}}{(2ax + x^2)}$, and $T = \left(\dfrac{L}{4m}\right)^{\frac{1}{2}} \times$ hyp. log. $a + x + (2ax + x^2)^{\frac{1}{2}} + $ corr. Let $T = 0$, $x = 0$, and corr. $= -\left(\dfrac{L}{4m}\right)^{\frac{1}{2}} \times$ hyp. log. a; $\therefore T = \left(\dfrac{L}{4m}\right)^{\frac{1}{2}} \times$ hyp. log. $\dfrac{a + x + (2ax + x^2)^{\frac{1}{2}}}{a}$. Now when the chain

quits the pulley, $x = \frac{1}{2}(L - a)$; $\therefore$ the whole time

$$= \left(\frac{L}{4m}\right)^{\frac{1}{2}} \times \text{hyp. log.} \; \frac{\frac{1}{2}L + \left((aL - 2a^2) + (\frac{1}{2}L - a)^2\right)^{\frac{1}{2}}}{a}$$

$$= \left(\frac{L}{4m}\right)^{\frac{1}{2}} \times \text{hyp. log.} \; \frac{L + (L^2 - 4a^2)^{\frac{1}{2}}}{2a}.$$

PROB. '109.

Suppose a weight, suspended by a cord passing over a fixed pulley, to be uniformly drawn up; required the number of vibrations which the weight would make before it reaches the pulley.

Let $a =$ the distance of the weight from the pulley at the beginning of its motion, $x =$ any variable distance ascended, and $v =$ the space through which it is uniformly drawn up in $1''$; $m = 16\frac{1}{12}$ feet; then the length of the pendulum at the distance $x = a - x$; therefore, time of oscillation : $\left(\frac{a - x}{2m}\right)^{\frac{1}{2}}$ :: circumference of a circle : diameter; therefore the time of one oscillation $= \dfrac{C . (a - x)^{\frac{1}{2}}}{D . (2m)^{\frac{1}{2}}}$; hence the number of oscillations in a time $= \dfrac{\dot{x}}{v}$, or the fluxion of the number of oscillations, whilst the weight ascends through the space x, $= \dfrac{D . (2m)^{\frac{1}{2}} . \dot{x}}{C \times V . (a - x)^{\frac{1}{2}}}$

$= \dfrac{(2m)^{\frac{1}{2}} . D}{C \times V} \times \dfrac{\dot{x}}{(a - x)^{\frac{1}{2}}}$, whose fluent, when $x = a$, is
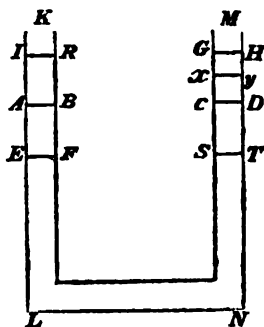
$\dfrac{2a^{\frac{1}{2}} . (2m)^{\frac{1}{2}} . D}{C \times V} =$ the number of vibrations made by the pendulum.

Cor. The number of vibrations, made by this pendulum, = twice the number made in the same time, viz. $\frac{a}{v}$, by a common pendulum, whose length is a.

Prob. 110.

Let *KLNM* represent a cylindrical tube, whose arms *KL* and *NM* are perpendicular, and *LN* parallel to the horizon; suppose it to be partly filled with water, and the surface of the water, when quiescent, to coincide with *AB* and *CD*; let *AB* be depressed to *EF*, and *CD* raised equally to *GH*; draw *IR* in the other arm parallel to *LN*, and at the same height with *GH*; then, the pressure being removed, *GH* will descend, and *EF* ascend. Required the time in which *EF* will rise to its greatest altitude *IR*.

Draw *ST* in *MN* at the same altitude with *EF*, and parallel to *LN*. Let $GC = a$; then $GS = 2a$, and the force with which the upper surface endeavours to descend = a column of the fluid, whose altitude is $2a$. Let the axis of the tube $= L$, and the force with which gravity would accelerate the fluid, if unconfined, $= 1$; then the column GS : the whole weight of the fluid :: $2a$: L; ∴ the force of this column $= \frac{2a}{L}$. Let *GH* descend to xy, and put $Gx = x$; then the accelerating force $= \frac{2a - 2x}{L}$; hence, if z be the space through which a body would fall by gravity to acquire the velocity at xy, $\dot{z} = \frac{2a\dot{x} - 2x\dot{x}}{L}$,

and $x = \dfrac{2ax - x^2}{L}$; $\therefore$ the velocity at $x = \left(\dfrac{4m}{L}\right)^{\frac{1}{2}} \cdot (2ax - x^2)^{\frac{1}{2}}$,

if $m = 16\frac{1}{16}$ feet. Hence $\dot{T} = \dfrac{\dot{x}}{v} = \left(\dfrac{L}{4m}\right)^{\frac{1}{2}} \times \dfrac{\dot{x}}{(2ax - x^2)^{\frac{1}{2}}}$;

and $T = \left(\dfrac{L}{4ma^2}\right)^{\frac{1}{2}}$. a circular arc of radius a, and versed

sine x, which needs no correction; $\therefore$ the whole time of

ascent to IR, when $x = 2a$, is $\left(\dfrac{L}{4ma^2}\right)^{\frac{1}{2}} \times \frac{1}{2}$ the circum-

ference of a circle of radius $a = \left(\dfrac{L}{4ma^2}\right)^{\frac{1}{2}} \times pa$ (if $p =$

3.14159, &c.) $= p \times \left(\dfrac{L}{4m}\right)^{\frac{1}{2}}$.

Cor. 1. The time of one descent of the fluid from GH to ST is equal to the time in which a pendulum would vibrate, whose length is half the length of the tube's axis.

Cor. 2. Since HD does not enter into the expression, whatever be the altitude of GH, the whole time of descent will be the same.

Cor. 3. The velocity of the descending surface is accelerated to CD, and then retarded to ST, where it $= 0$.

Prob. 111.

Let the arms of the tube be inclined to the horizontal part LN in any given angles, and let AB be depressed as before to EF; to find the time, in which, if the pressure were removed, EF would rise to its greatest altitude IR.

Let AI or AE, which are equal to HD and DT, $= a$; $p =$ the sine of the angle KLN, $q =$ the sine of the angle MNL, to a radius 1. Let $L =$ the length
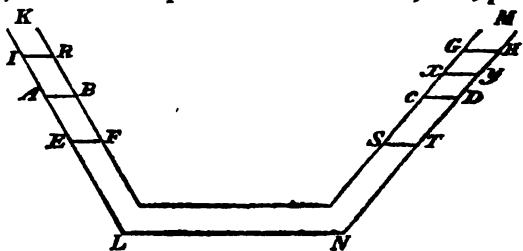
of the whole canal, and the force with which gravity would accelerate the fluid if unconfined $= 1$. Then the force, by which the column HD endeavours to descend at first $=$

$$\frac{HD \times \text{the sine of } HNL}{r} + \frac{AE \times \text{sine of } KLN}{r} = HD \cdot$$

$(q+p) = a \cdot (q+p) =$ the weight of the two columns HD and AE in a perpendicular direction. Hence the accelerating force at $H = \dfrac{a \cdot (q+p)}{L}$; and when the surface GH has descended to xy, if $HY = x$, the accelerating force on

$$xy = a \times \frac{(q+p)}{L} - x \times \frac{(q+p)}{L} = \frac{(a-x) \cdot (q+p)}{L} \text{; and if } z$$

represent the space through which a body must fall freely by gravity to acquire the vel. at xy, $\dot{z} = \dfrac{(a\dot{x} - x\dot{x}) \cdot (q+p)}{L}$,

and $z = \dfrac{(2ax - x^2) \cdot (q+p)}{2L}$; hence the velocity at $y =$

$\left(\dfrac{(q+p)}{2L} \cdot 4m\right)^{\frac{1}{2}} \cdot (2ax - x^2)^{\frac{1}{2}}$. Now $\dot{T} = \dfrac{\dot{x}}{v}$; $\therefore$ in this

case, $T = \left(\dfrac{2L}{4m \cdot (q+p)}\right)^{\frac{1}{2}} \times f \cdot \dfrac{\dot{x}}{(2ax - x^2)^{\frac{1}{2}}}$, and $T =$

$\left(\dfrac{2L}{4ma^2 \cdot (q+p)}\right)^{\frac{1}{2}} \times$ a circular arc of radius a, and versed sine x, which needs no correction. Let $x = 2a$; then the

whole time $= \left(\dfrac{2L}{4ma^2 \cdot (q+p)}\right)^{\frac{1}{2}} \times \frac{1}{2}$ the circumference of a

circle of radius $a = \left(\dfrac{2L}{4ma^2 \cdot (q+p)}\right)^{\frac{1}{2}} \cdot Pa$ (if $P = 3.14159$,

&c.) $= P \cdot \left(\dfrac{L}{2m \cdot (q+p)}\right)^{\frac{1}{2}}$.

COR. 1. If KL and MN are perpendicular to the horizon, or p and q each $= 1$, $T = P \cdot \left(\dfrac{L}{4m}\right)^{\frac{1}{2}}$, as before.

Cor. 2.   The time of one descent of the surface from
GH to *ST*, or ascent from *EF* to *IR*, is equal to the
time in which a pendulum would vibrate, whose length is
$$\frac{L}{p+q}.$$

Cor. 3.   When the angles at *L* and *N* each equal 30°,
then *p* and *q* each $= \frac{1}{2}$; and the length of a pendulum,
which would vibrate in the same time, $= L$.

PROB. 112.

The force of gravity being supposed constant, to find
the density of the air at any altitude above the surface of
the earth.

Let $r =$ the radius of the earth, $x =$ any variable distance
from the center above the surface, $v =$ the density of the
air at that point, the density at the surface being assumed
$= 1$; $h =$ the height of an homogeneous atmosphere.
Now since the compressive force of the air is proportional
to its density, the fluxion of the compressive force is to
the fluxion of the density as the compressive force to the
density; and in this case, since the force is constant, the
fluxion of the compressive force is as the density and the
fluxion of the altitude; hence $h : 1 :: v\dot{x} : -\dot{v}$; $\therefore$
$- h \times \dfrac{\dot{v}}{v} = \dot{x}$, and $- h \times$ hyp. log. $v + \text{corr.} = x + \text{corr.}$, or

$h \times$ hyp. log. $\dfrac{1}{v} = x - r.$

Cor. 1.   If x, and therefore $x - r$, be assumed in
arithmetic progression, the hyp. log. of $\dfrac{1}{v}$ is in arith-
metic progression, and $\dfrac{1}{v}$ in geometric; that is, the den-
sities of the air are in geometric progression.

Cor. 2. If *AB* represent the den-
sity of the air at the earth's surface,
and *AH* be drawn perpendicular to
AB, and any ordinates, as *FG*, be
taken proportional to the density at *E*
F, the curve *BGI* thus traced out is
the logarithmic curve.
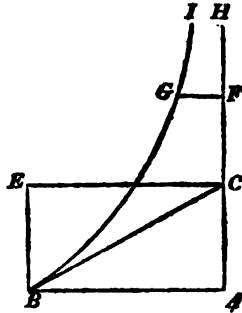
For $x - r \propto$ hyp. log. $\dfrac{1}{v}$, or *AF* varies inversely as the

logarithm of the density; hence, if *AF* increases in arith-
metic progression, *FG* decreases in geometric, a property
of the logarithmic curve. (Art. 131.)

Cor. 3. Let $h =$ the height of a homogeneous atmo-
sphere, in which the density throughout is the same with
the density at *A*; the pressure at *A* is the same, whether
we take the homogeneous atmosphere, or that whose
density is represented by the varying line *FG*. Hence
$h \times AB =$ the area $ABIH = AB \times AC$ (Art. 133. Prob. 3.
Cor. 1.); $\therefore$, $h = AC =$ the sub-tangent, or the modulus of
this system of logarithms is the height of an homogeneous
atmosphere. (Art. 132. Prob. 2. Cor. 2.) Cotes' *Harm.*
Mens. Prop. 5. Scholium.

PROB. 113.

The force of gravity varying inversely as the square of
the distance from the center of the earth; to find the
density of the air at any altitude above the surface.

The force of gravity at any distance $x = \dfrac{r^2}{x^2}$. Now the

compressive force of the air varies as its density; and the
fluxion of the compressive force is proportional to the
force of gravity, the density and the fluxion of the altitude;
hence, on the same assumption as in the last Problem,

$$h : 1 :: \frac{v r^2 \dot{x}}{x^2} : -\dot{v}; \ \therefore h \times -\frac{\dot{v}}{v} = \frac{r^2 \dot{x}}{x^2},$$

and $- h \times$ hyp. log. $v = -\dfrac{r^2}{v} +$ corr. ; that is, $h \times$ hyp. log.

$$\frac{1}{v} = r - \frac{r^2}{x} \, .$$

Cor. 1. If x increase in musical progression, $\dfrac{r^2}{x}$ is in arithmetic ; hence the hyp. log. of the densities decreases in arithmetic progression, and the densities themselves in geometric.

COTES' SOLUTION. HARMONIA MENSURARUM.

Prop. 5. Scholium.

Let S be the center of the earth, AB its surface. As-sume $Sf : SA :: SA : SF$, and draw the ordinate fg proportional to the density at F; the curve Bgn traced out by the point g will be the loga-rithmic curve, but in an inverted position.   For let AF be increased by the small line FM; take $Sm : SA :: SA : SM$, and assume mn, an ordinate proportional to the den-sity at M.   Then, since $Sm \times SM = SA^2 = Sf \times SF$, we have $Sm : Sf :: SF : SM$; and $dividendo$, $Sf - Sm : Sf :: SM - SF : SM$; $alternando, fm : FM :: Sf : SM ::$ $Sf : SF :: SA^2 : SF^2$; $\therefore fm =$
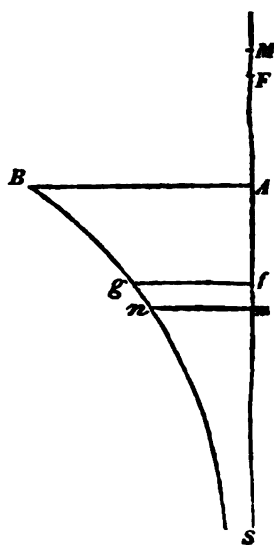
$\dfrac{FM \times SA^2}{SF^2}$; hence $fm \times fg$ varies as the fluxion of the distance $\times$ the force of gravity at $F \times$ the density, or the area $fgnm$ varies as the pressure at F; and the sum of all the similar areas below fg varies as the sum of the pressures above F, or varies as the density fg. In the same manner the sum of all the areas below mn varies as mn; hence their difference, or the area $fgnm, \propto fg - nm$; that is, $fg \times fm \propto fg - mn$; $\therefore$ if $fg = y$, and $fm = \dot{x}$, and $fg - mn = \dot{y}$, $y \times \dot{x} \propto \dot{y}$, or $\dot{x} \propto \dfrac{\dot{y}}{y}$, a property of the logarithmic curve. This is curve is the same with that in the last Proposition; for the ordinates very near AB, and at very small equal intervals, are in each case equal; hence, in both cases, the curvature, the inclination of the tangent at B, and the value of the sub-tangent, are the same.

Prob. 114.

The force of gravity being supposed to vary as the nth power of the distance from the center of the earth, and the compressive force of the air to be proportional to its density; to find the density of the air at any altitude above the surface.

Here the force of gravity, at a distance x from the center, $= \dfrac{x^n}{r^n}$; therefore, on the same assumption as in the preceding case, $h : 1 :: \dfrac{v x^n \dot{x}}{r^n} : -\dot{v}$; $\therefore -h \times \dfrac{\dot{v}}{v} = \dfrac{x^n \dot{x}}{r^n}$, and

$-h \times$ hyp. log. $v + \text{corr.} = \dfrac{x^{n+1}}{(n+1) . r^n} + \text{corr.}$ Now if

$x = r,\ v = 1$; therefore $h \times$ hyp. log. $\dfrac{1}{v} = \dfrac{x^{n+1} - r^{n+1}}{(n+1) . r^n}$.

Cor. 1. If the force be constant, $n = 0$, and $h \times$ hyp. log. $\frac{1}{v} = x - r$, as in Prob. 111.

Cor. 2. If the force $\propto \frac{1}{D^2}$ from the center, $n = -2$, and $h \times$ hyp. log. $\frac{1}{v} \propto 1 - \frac{r}{x}$, as in Prob. 112.

COTES' SOLUTION.

Let S be the center of the earth, A a point on its surface, $SAF2$ a line drawn from the center to the top of the atmosphere; and let it be required to find the ratio of the density in A to the density in F, the force of gravity varying as SF^n. Let $SF = x$, $d =$ the density at A, and v the density at F; then, since the compressive force varies as the density, the fluxion of the density is as the fluxion of the compressive force, or $\dot{v} \propto v x^n \dot{x}$, and $\frac{\dot{v}}{v} \propto x^n \dot{x}$.

Let AC be the height of an homogeneous atmosphere; that is, of an atmosphere, whose density throughout is the same with the density at A; then AC : the height of the mercury in a barometer at A :: the density of mercury : the density of the air at A; and if F be conceived to approach toward A, the altitude of the mercury in the barometer at A : its altitude at F :: AC : FC. Hence the density of the air (d) at A : its density (v) at F :: AC : FC; $\therefore d - v$, or $\dot{v}$: d, or v ::
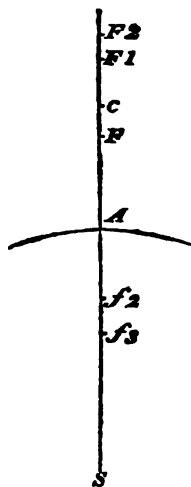
AF, or $\dot{x} : AC$; whence, in this case, $AC \times \dfrac{\dot{v}}{v} = \dot{x} = \dfrac{x^n \dot{x}}{SA^n}$.

Since then, wherever F be assumed, $\dfrac{\dot{v}}{v} \propto x^n \dot{x}$, $AC \times \dfrac{\dot{v}}{v}$ will

$= \dfrac{x^n \dot{x}}{SA^n}$ in all cases, whatever be the position of F.

EXAMPLES.

Ex. 1. Let the force vary as $\dfrac{1}{D}$ from the center.

Here $n = -1$, and $AC \times \dfrac{\dot{v}}{v} = SA \times \dfrac{\dot{x}}{x}$; $\therefore$ $AC \times$ hyp.

log. v + corr. $= SA \times$ hyp. log. x + corr.; that is, since $\dot{v}$
is negative when $\dot{x}$ is positive, and the density at $A = d$,

$$AC \times \text{hyp. log. } \frac{d}{v} = SA \times \text{hyp. log. } \frac{x}{r},$$

or the measure of the ratios between the densities d and
v to the modulus AC = the measure of the ratios between
SF and SA to the modulus SA.

Ex. 2. Let the force of gravity $\propto D^n$ from the center.
In this case, by taking and correcting the fluents,

$$AC \times \text{hyp. log. } \frac{d}{v} = \frac{1}{n+1} \times : \frac{x^{n+1}}{SA^n} - SA,$$

the measure of the ratio between the densities d and v, to
the modulus AC.

Assume SA, SF, $SF1$, $SF2$, in geometric progression
increasing; and SF, SA, $Sf2$, $Sf3$, in the same pro-
gression decreasing; and let the force of gravity vary as
$\dfrac{1}{D^3}$; the measure of the ratio between the densities at A

and F, to the modulus AC, or $AC \times$ hyp. log. $\dfrac{d}{v} = \frac{1}{2}$

$SA - \frac{1}{2} \dfrac{SA^3}{SF^2}$.

Now, since $SF : SA :: SA : Sf2$,

and $SA : Sf2$ (or $SF : SA$) $:: Sf2 : Sf3$,

we have, by the first proportion,

$$SF : Sf2 :: SF^2 : SA^2;$$

and by the second, $Sf2 : Sf3 :: SF : SA$;

$$\therefore SF : Sf3 :: SF^3 : SA^3;$$

hence $Sf3 = \dfrac{SA^3}{SF^2}$, and $\frac{1}{2}SA - \frac{1}{2}\dfrac{SA^3}{SF^2} = \frac{1}{2}SA - \frac{1}{2}Sf3 = \frac{1}{2}Af3$;

$\therefore AC \times$ hyp. log. $\dfrac{d}{v} = \frac{1}{2}Af3$. In the same manner, if

the force of gravity $\propto \dfrac{1}{D^2}$, $AC \times$ hyp. log. $\dfrac{d}{v} = Af2$; if

gravity be uniform, it $= AF$; if gravity vary directly as

the distance, it $= \frac{1}{2}AF1$; if gravity vary as D^2, it $= \frac{1}{3}AF2$,

and so on, in infinitum.